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MODAL LOGIC AND THE "POSSIBLE"

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Abstract

If we accept the idea that the value of a "logic" depends on its ability to fit the usual (ordinary) meaning of the "logical constants" which it formalises and the inferences we draw on their basis, we may ask if the meaning of "possible" is well captured by normal modal logic. We are faced with the following puzzle: if ϕ is meaningful and non contradictory is it not a logical truth that $\Diamond\phi$? First, it is shown why there is no logical law of the form $\Diamond\phi$, ϕ being a neutral wff, and then some philosophical conclusions are drawn: "possible" relative to truth must be distinguished from possible "relative" to meaning, and the former only is formalised in modal logic.

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As it is well known, modern logic at its beginnings rejected, as extra-logical, modal notions like "necessary" or "possible". In Russell's view, the idea that logic has to give account of such notions, is the result of an erroneous conception of the truth. For him, a proposition is merely true or merely false and to say that it could be "necessarily" (or "possibly") true makes sense only if we accept the subject-predicate view: a necessary truth is one which attributes to a given subject what is traditionally called an "essential" property. The distinction between necessary truth and mere truth is the opposition between what is grounded on the very "essence" of the subject, so that the proposition expresses what *is* the subject, and a mere truth which expresses a "simple fact", that is to say what happens to be the case of the subject but could have not been the case. A necessary truth is one which, being firmly grounded in the nature or essence of the subject, is, at the end, logically grounded on the Principle of Contradiction: if a given subject did not possess such and such "essential" properties, it would be different from itself, while a proposition which states a "simple" fact, states what the case is of a subject but which could have been different without the subject being affected. In this traditional line of thought, if something like a Principle of Sufficient Reason is admitted, it is impossible to admit relational propositions not reducible to subject-predicate propositions, because that would amount to admitting something lacking “reason”[1].

In a paper written in 1905 for a conference at the Oxford Philosophical Society, Russell distinguishes four different meanings of “necessary”: a proposition is necessary (i) if it is known *a priori*, (ii) if it has been proved, (iii) if it has been proved on purely logical grounds (i.e. analytical truth), and (iv) ultimately we may say that a *propositional function* is necessary if it is true for every argument. Thus, there is nothing in the idea of necessity which is irreducible, and Russell concludes that there “is no one fundamental logical notion of necessity, nor consequently of possibility”. Indeed, “the subject of modality ought to be banished from logic, since propositions are simply true or false, and there is no such comparative and superlative of truth as is implied by the notions of contingency and necessity.” [2]

The difficulty

What about modal logic then? A customary view about what is the aim of a “logic” may be stated as follow: within our (ordinary) languages there are words or phrases on the “meaning” of which are based valid inferences, the so-called “logical constants”. The most famous of them are the well known propositional connectives and the quantifiers of first order logic. But we have to recognise that valid inferences are drawn which are based on the meaning of other phrases, for example on the meaning of such words as “necessary” or “possible” (e. g. necessarily (p and q), iff necessarily p and necessarily q). Thus, what has been done for the former, can be done for the later; hence (alethic, and others) modal logic.

What is the “meaning” of a logical constant? In the case of propositional connectives and quantifiers, the answer is given by the standard evaluation rules and this amounts to stating the contribution of these little words to the truth value of a complete sentence in which they have an occurrence. The result is perhaps that our “intuition” as to what is the meaning of “and” or of “there is” is not well translated, but it is not really a problem because what is given is an *a priori* account of

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[1] Russell’s rejection of modalities is then closely bound with his engagement for the thesis of “external relations”.

[2] See Russell [8], p. 520.

these logical constants, independently of what *we* think their meaning is in our ordinary languages. So these evaluation rules are not really a (disputable) formal translation of some "meaning intuition". The point is simply this: given n propositions there are $2^{(2^n)}$ truth functions, no more no less, regardless of our ordinary connectives; given a propositional function, it can be true for all or at least one of its arguments and this means that the corresponding logical product (resp. logical sum) is true. "Meaning" is reduced to "truth" (extensionalism) and logic has to do with truth, not with meaning[3].

Clearly, the situation is much less favourable as soon as we leave this comfortable home and have to deal with intensional operators like "necessary" or "possible". Our purpose in this paper is to consider a feature of "possible" which seems to resist (usual) modal logical treatment, and to draw some conclusions on this point.

This feature is this: given a well-formed non contradictory proposition p, is it not a logical truth that $\Diamond p$? From "my black rabbit is born on the hidden face of the Moon" is it not valid to infer "it is possible that my black rabbit be born on the hidden face of the Moon"? Clearly, we can give a positive answer, relying on ordinary behaviour of French (or Spanish or Polish) speakers: if I assert that my black rabbit is born on the hidden face of the Moon and if my friend does not agree, at least, if I ask him to do so, he is compelled to agree with the fact that it is *logically* possible for such a rabbit to be born on such a place even if he denies that it is physically or biologically possible. Against this, as it is well known, in normal modal logics a formula of the form $\Diamond\phi$ (with ϕ non valid and non contradictory (neutral) wff of propositional (PC) or first order logic[4]) is never a logical truth. It is not uninteresting to see why.

Of course we have from the T-system on, the logical law:

$$\vdash \phi \quad \Diamond\phi$$

which is a direct consequence of the T-axiom:

$$\vdash \Box\phi \quad \phi$$

But first, it has to be noticed that what this law states is only that if a proposition ϕ is true then it is true that it is possible that ϕ. And it is not what we want for we want to assert "it is possible that ϕ" even if it

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[3] In fact, this assertion is the hidden core of this little paper.

[4] In the following we will consider only modal propositional logic for the sake of simplicity and because what we will try to suggest applies as well to both "logics".

is not true that ϕ (and even if all we know forbids us to think it be true, but this is only an epistemological accident).

Moreover, it is not what we want because if we wish to obtain

⊢ ◇ϕ

we have to apply Modus Ponens on this formula and so we must first deduce

⊢ ϕ

so that ϕ ought to be a logical law, and obviously this is not the case of any PC-neutral proposition (as that about my black rabbit). That is not to say that in modal logical systems we have no law of the form ◇ϕ where ϕ is not a *modal* logical law[5]: in fact, as it is well known, we have, for example, the law '◇(p □p)' in T, and clearly 'p □p' is not, in any system, a modal logical law. But it is impossible to have ϕ a neutral *PC*-wff. This can be shown very simply. Every PC-transform of a modal logical law (i.e. the PC-wff obtained from a MPC-wff by deleting all modal operators) is a PC-tautology; but if we have ◇ϕ, ϕ being a neutral PC-wff, its PC-transform is ϕ itself, which is not a tautology, and so ◇ϕ is not a modal logical law.

We have then to admit, on the basis of modal logic, that valid (tautologous) formulae of elementary logic alone are possible (but as they are, in modal logic, necessary, it is not very upsetting!) and that it is never a logical law that a formula which could be true, but is not always true, is "possibly" true.

If we turn to semantical considerations, we meet the following situation. A formula of the form ◇ϕ is S5-valid[6] iff in every world of every S5 model (i.e. every model founded on reflexive/symmetric/transitive frames), ◇ϕ is true. It is very easy to devise a model M_o: <W, R, V>, such that a formula ◇ϕ (with ϕ neutral PC-wff), will not be valid, i.e. in which ◇ϕ will be false in at least one world (in fact in all worlds of a S5-model). For the sake of simplicity, let us take R such that every world is related to every world; we define V such that for every w W, V(ϕ,w)=f (which is possible since ϕ is not PC-valid); in none of the M_o's worlds ◇ϕ can be true, so ◇ϕ is obviously not valid in M_o and *a fortiori* is not S5-valid. On the other hand, if a formula of the

[5] Apart from systems K (where there is no law of the form ◇ϕ) and D where, in fact ◇ϕ can not be a law without ϕ being equally a law. What we say holds for system T and all the stronger systems.

[6] We will confine ourselves to this kind of model since if what we will say holds for S5-models, it will hold for others.

form $\Diamond\phi$ is S5-valid, this means, in particular, that it is valid in every S5-one-world-model (i.e. with $W=\{w_i\}$); if so $\Diamond\phi$ is valid only if ϕ is true in each world, i.e. only if ϕ is PC-valid[7].

If we come back to the first approach, we are in the following situation: if we want $\Diamond\phi$ logically true ϕ being a neutral PC-wff, we should have to admit something very curious: we can infer from each ϕ PC-wff and non contradictory, all the theorems of the form $\Diamond\phi$, which amounts to add a very unusual rule of inference and, to change strongly the substitution rule for, as it is well known, there are contradictory substitutional instances of a neutral PC-wff; so that, unless we make some change, we should have to accept as a law $\Diamond\phi$ with ϕ contradictory (or more precisely with $\neg\phi$ deductible)! All this seems hardly manageable.

What kind of semantic would fit our requirement? Something like that[8]: in PC, we define a truth set Σ as a set of all the formulae true under a given evaluation (i.e. for a given evaluation V, the corresponding truth set Σ is the set of all ϕ such that $V(\phi)=\text{T}$). We construct a model M_o for modal logic as the set of all truth sets (which play the role of the traditional worlds so that 'ϕ is true in w' = 'ϕ w') and we forget all kind of "accessibility relation", that is to say we let every world (i. e. truth set) "see" every world (God's point of view). Our evaluation rules for modal formulae are quite obvious for the "propositional" part and for the modal part they look like that:

$M_o \vDash \Box\phi$ iff for every world w M_o, ϕ w
$M_o \vDash \Diamond\phi$ iff for at least one world w M_o, ϕ w.

Clearly, every formula $\Box\phi$, with ϕ PC-valid, is valid in M_o (since the PC-valid formulae are just those formulae which belong to all the truth sets), and all the axioms of K, D, T, S4, S5 are valid. In addition, all the formulae of the form $\Diamond\phi$, ϕ being a neutral PC-wff, are valid in M_o for each such formula belongs to at least one truth set, that is to at least one "world". It must be noticed that there is no question of "frame": we allow only *one* model because we want to capture the idea that if a formula is not contradictory but only well formed, we can find, from our divine point of view, a "world" in which it is true. So we have to rule out models in which a non contradictory, but non tautological, formula is

[7] We take this last argument from H. Field [2], p. 117. All these considerations hold a fortiori for systems weaker than S5.

[8] The following is not very far from Carnap's modal semantics in [1], Ch. V.

never true (in S5 models), for this was the very root of our difficulties: as we noticed, in the kripkean perspective, it is always possible to construct a model in which the possibility of a wff of this kind is not valid because the evaluation V which characterises the model has "forgotten" to give value "true" to it in at least one world. And there is no other way for a formula $\Diamond\phi$ to be valid than ϕ being true (in S5-models) in at least one world. But this does not mean that in another model our formula ϕ would not be true in a world, and it is precisely for this that kripkean semantic fails to give a satisfactory account of "possible": a formula $\Diamond\phi$ is valid in a S5-model if it is possible, in each world, for ϕ to be true, that is to say, true in another accessible world, but nothing can guarantee that, if it "can" be true "in the absolute", it is *in fact* true in at least one world of *this* model; and here "in the absolute" means, "in at least one world in *some* model". In short, we have two senses of "possible": one which depends on whether a formula takes value true in at least *one* world in *one* model, and another which depends on whether a formula takes value true in *some* world of *some* model. We can evaluate a $\Diamond$-formula in each world of a model, but this evaluation does not take into account what happens in some other model and it is just this last requirement that we need. In other words, to fulfil our requirement that it is logically true that a mere non contradictory PC-wff is possible, it seems that we would have to admit that for $\Diamond$-formulae, truth in a world of a model collapses into S5-validity. It is this requirement that our little semantics fulfils for we have no more than one model and, in it, we have put all the possible "worlds", that is to say: if a formula can be true, i.e. if there is (in the absolute) an evaluation which gives it value true, then we are sure that we have in the model the corresponding world (truth set).

But at what price? Clearly, with all this we say nothing more that 'ϕ is tautologous' or 'ϕ is neutral' so that our modal operators have, in fact, disappeared! In short we have lost all what modal logic is supposed to bring us.

Some further philosophical and historical remarks

One way of expressing our worries is to say that "possible" and "necessary" are not on the same level: "Necessary" has to do with *truth*, "possible" with *meaning*. Of course, it seems reasonable enough to say that $\Box\phi$ is true if ϕ is always true, $\Diamond\phi$ is true if ϕ is sometimes true, and it is on this ground that modal logic can "reduce" modalities to truth and can construct a corresponding semantics. But is it not a false symmetry?

Of course if a proposition is necessary then it must be always true, but what about possible? Clearly, if a proposition is once true then it is possible. But is it necessary that if a proposition is possible it must be sometimes true? What is implied in what we previously said is that we don't need any reference to truth or falsehood to establish that "it is possible that…"; what we need is to *understand* the proposition; and this is necessarily done before we can enquire about truth and falsehood, since we could not ask about truth or falsehood if we did not understand about what we are asking[9]. In other words, we have first to grasp the meaning of the proposition, and it is on this basis that we can say that "it is possible that…". So possibility is directly inferred from meaningfulness and it is this feature which led us to the deviant rule: from a non contradictory wff ϕ infer $\Diamond\phi$. That is not to say that we must *reject* the transition from "possible" to "sometimes true", but that this "principle"[10] is missing the point, or, more accurately, that, at this stage, we have neither to assume nor to reject it for, to repeat, we have first to assume *possibility* before any question about truth or falsehood. We could resume this point by: ϕ is possible = ϕ has a meaning.

On the other hand, it seems more natural to treat "necessity" in terms of truth, and in fact the semantics of modal logic doesn't present the same discrepancy as in the case of the "possible". It has been argued sometimes[11] that the kripkean definition of "necessity in a world" is too narrow because what we want for ϕ being necessary in a world is not only ϕ being true in all (accessible) worlds of a model, but ϕ being true in any world. But this is not so harmful because, in the case of S5-valid $\Box$-formulae, we have to take account of any S5-model, so that we have to take into account all the worlds; and, as it is not the merely true (in a world) $\Box$-formulae which are of interest for us but the S5-valid ones, we can meet this objection. But the parallel with the case of the $\Diamond$-formulae is interesting: if in a S5-model there is no ϕ-world (i.e. world in which ϕ is given the value true by the evaluation V), $\Diamond\phi$ is false and so not S5-valid; but that does not mean that there is no ϕ-world in some other model and if so, we would wish our semantic let $\Diamond\phi$ be valid. For

9 It ought to be clear that this point of view (roughly wittgensteinian) is highly disputable but it is not the place here to discuss it.

10 This "principle" is akin to the old principle that what is possible will be realized (or : what is never realized, is impossible) which is so stated by Arthur Lovejoy [6], p. 52 : "…no genuine possibility of being can remain unfulfilled…". For a discussion of this "principle" in Aristotle, see J. Hintikka [3], especially, Ch. V.

11 See, for example, I. Niiniluoto & E. Saarinen [7].

□-formulae the situation is not the same. Clearly if all worlds of a S5-model are ϕ-worlds, □ϕ is valid in this model but it is not the end of the history and we have to continue our semantical journey, so that we will take into account *all* worlds in all models[12], and then the seemingly narrowness of the definition of "necessary in a world" is overcome. And it is just this last feature which is lacking in the case of ◇-formulae and which causes the troubles.

It could be argued that "necessary" is more than "always true", but this "more" is probably of a metaphysical and ineffable nature and if modal logic, via "always true" captures the essential feature of the necessity operator, we can be content with it.

We arrive at our little historical remark. As it is well known modern modal logic arose from the troubles that the use of material implication by Russell (and Frege) caused to C.I. Lewis. If, as Russell put it, the study of material implication (the so-called "Theory of Deduction" of PM) is the study of the relation that two propositions must have for one being deductible from the other, we are faced with the classical "paradoxes" of material implication. Lewis thought that the trouble came from the fact that, in a "true" material implication, it is not necessary for the antecedent and the consequent to have a *link*, so that both propositions may be independent from each other. Obviously this idea of a link is not quite clear[13] and to give it more substance, Lewis introduced modalities and defined strict implication by □(ϕ   ψ)[14]. It was then supposed that "necessity" was clearer than this vague idea of a link between propositions. In fact, the idea of necessity is not so clear, but it soon appears that the idea of *logical* necessity is (extensionally) easily defined as the set of all logical laws (which is recursive in the case of the

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[12] In modal predicate logic this notion is not so clear…

[13] In Lewis' perspective. But in the case of the theorems of the "Theory of Deduction" or of any system for elementary logic, this "link" is just the "interpolate" of the two propositions : if 'ϕ   ψ' is a logical law in which neither ¬ϕ nor ψ are logical laws, then there are some propositional variables which are common to ϕ and ψ, and we can form a new formula θ out of these propositional variables alone, such that ϕ   θ and θ   ψ are logical laws. Clearly we can think of θ as this somewhat mysterious link. As Wittgenstein put it at the same time, it is not "ϕ   ψ" which expresses deductibility but "ϕ   ψ being a tautology", and the necessity operator in □(ϕ   ψ) may be understood as expressing no more than "is a tautology".

[14] More precisely, Lewis defined strict implication by : ¬◇(ϕ   ¬ψ) which amounts to the same : "A strict implication is equivalent to a material implication which holds of necessity, or by purely logical analysis". [5] p. 165.

propositional logical laws and recursively enumerable in the case of the first order logical laws). This story looks like a circle: one starts from the idea that in a "Theory of Deduction" material implication does not capture the idea of deductibility, then "necessity" is introduced with the aim of strengthening this too weak relation and to clarify the vague idea of a link between propositions, but, at the end, one comes back to the old "Theory of Deduction" to which it is asked to answer the question: what is (logical) necessity? So, good for "logical necessity", but what about the "possible"? Defined as $\neg\Box\neg$, it seems unproblematic, but, as we saw, at the cost of lacking an important feature of what we should require of a "logic" of the possible. And it is not so surprising: in classical elementary logic, we are concerned with truth and falsehood because, in the ultimate, we are trying to show that certain propositions of a given "form" are never false when others of a given form are true, and this can be done by showing that the corresponding implication is always true, true in all "possible" worlds, necessarily true, etc., etc. In the case of the possible, besides the possibility of what is sometimes true, we want, in addition the possibility of what is meaningful, and, with this, we leave our comfortable tarskian-kripkean universe.

The moral of this story could be expressed in a somewhat Aristotelian fashion: there are two senses of "possible", possible relative to truth / possible relative to meaning. The first is more or less captured by normal modal logic (S5), not the second, and this amounts to say that modal logic fits rather well the necessary, but not the possible.

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