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KRIPKE SEMANTICS FOR SOME PARACONSISTENT LOGICS

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Abstract

The paper deals with seven propositional paraconsistent logics. Four of them are based on intuitionistic positive logic: the minimal Johansson's logic, some two its weakenings and its extension by the law of excluded middle. The remaining three ones are their counterparts having the classical positive base. For all logics the Kripke-style semantics is provided.

1. Introduction

The paper provides possible worlds semantics for a few propositional paraconsistent logics related to the minimal Johansson's logic [6]. Originally this logic arose by removing the Duns Scotus' thesis $\neg\alpha \to (\alpha \to \beta)$ from the Heyting's set of axioms defining the propositional intuitionistic logic [5], i.e., it was defined by means of a set of axioms of Hilbert's positive logic (HPL) (the positive part of intuitionistic logic), the rule of *Modus ponens* (MP) and the single axiom involving the connective of negation "¬":

(J) $((\alpha \to \beta) \quad (\alpha \to \neg\beta)) \to \neg\alpha$.

We show that on the ground of (HPL) and (MP), the axiom (J) is equivalent to each of the two groups of axioms, one composed of a single axiom:

(Ax) $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$,

and the other one composed of the two axioms:

(Ax1) $\neg\alpha \to (\alpha \to \neg\beta)$,
(Ax2) $(\alpha \to \neg\alpha) \to \neg\alpha$.

(J) (Ax): The following sequence of formulas
$\alpha \to \neg\beta$, β, $\beta \to (\alpha \to \beta)$ (HPL), $\alpha \to \beta$ (MP), $(\alpha \to \beta) \quad (\alpha \to \neg\beta)$ (HPL)+(MP), $((\alpha \to \beta) \quad (\alpha \to \neg\beta)) \to \neg\alpha$ (J), $\neg\alpha$ (MP)
yields $\{\alpha \to \neg\beta, \beta\} \vdash_{HPL+MP+J} \neg\alpha$, so due to deduction theorem (Ax) follows.

(Ax) (Ax1): $\neg\alpha \to (\beta \to \neg\alpha)$ (HPL), $(\beta \to \neg\alpha) \to (\alpha \to \neg\beta)$ (Ax), $\neg\alpha \to (\alpha \to \neg\beta)$ (HPL)+(MP).

(Ax) (Ax2): The following sequence of formulas
$\alpha \to \neg\alpha$, $\neg\alpha \to (\alpha \to \neg(\alpha \to \alpha))$ (Ax1), $\alpha \to (\neg\alpha \to \neg(\alpha \to \alpha))$ (HPL)+(MP), $(\alpha \to (\neg\alpha \to \neg(\alpha \to \alpha))) \to ((\alpha \to \neg\alpha) \to (\alpha \to \neg(\alpha \to \alpha)))$ (HPL), $\alpha \to \neg(\alpha \to \alpha)$ (MP), $(\alpha \to \neg(\alpha \to \alpha)) \to ((\alpha \to \alpha) \to \neg\alpha)$ (Ax), $\alpha \to \alpha$ (HPL)+(MP), $\neg\alpha$ (MP)
yields $\{\alpha \to \neg\alpha\} \vdash_{HPL+MP+Ax} \neg\alpha$, so (Ax2) follows.

(Ax1) & (Ax2) (J): The following sequence of formulas
$\alpha \to \beta$, $\alpha \to \neg\beta$, α, $\neg\beta \to (\beta \to \neg\alpha)$ (Ax1), $\neg\alpha$ (MP)
shows that $\vdash_{HPL+MP+Ax1} (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to (\alpha \to \neg\alpha))$, so $\vdash_{HPL+MP+Ax1} ((\alpha \to \beta) \quad (\alpha \to \neg\beta)) \to (\alpha \to \neg\alpha)$, hence and from (HPL)+(MP), and (Ax2) the axiom (J) follows.

Let $\vdash_J$ denote the minimal Johansson's logic (as the consequence relation) defined on the standard propositional language (the connectives are , , $\to$, $\neg$) by the rules: (HPL), (MP), (Ax1) and (Ax2). We will show an adequate Kripke semantics for $\vdash_J$, for its strengthening by the axiom (EM) $\neg\alpha \quad \alpha$, and for two its weakenings by removing (Ax1) and (Ax2) respectively. The same will be done for the classical counterparts of those logics, that is, when the positive part of the logics is classical.

2. Semantics for Johansson's logic

Let M(J) be a class of all the Kripke models of the form M = <W,≤,A,ν>, where W ≠ , ≤ W^2 is reflexive and transitive, A W is any subset of W and ν : Var → P(W) is a function fulfilling the condition:

(*) p Var x,y W (x ≤ y (x ν(p) y ν(p))),
Var is the set of propositional variables.

Assume the truth conditions for positive formulas as in a model for Hilbert's positive logic ($x \models_M \alpha$ means that the formula α is true in M at the point x):

$x \models_M p$ iff x $\nu(p)$, p Var,
$x \models_M \alpha \quad \beta$ iff $x \models_M \alpha$ and $x \models_M \beta$,
$x \models_M \alpha \quad \beta$ iff $x \models_M \alpha$ or $x \models_M \beta$,
$x \models_M \alpha \to \beta$ iff y W $(x \leq y \quad (y \models_M \alpha \quad y \models_M \beta))$,

and the following one for "¬":

(¬) $x \models_M \neg\alpha$ iff y W $(x \leq y \quad (y \models_M \alpha \quad y \quad A))$.

Let $\models_{M(J)}$ be a consequence relation defined by the class M(J) in the standard way: for any set of formulas X and a formula α, $X \models_{M(J)} \alpha$ iff M M(J) x W ($x \models_M \alpha$ whenever for all β X, $x \models_M \beta$).

As in the case of intuitionistic logic, truth in a particular model M M(J) is preserved by its ordering $\leq$:

(**) M M(J) α x, y W $(x \leq y \quad (x \models_M \alpha \quad y \models_M \alpha))$,

therefore it is easy to show the soundness theorem: $\vdash_J \quad \models_{M(J)}$.

For the completeness proof let us apply the notion of prime theory of the logic $\vdash_J$: any theory X of $\vdash_J$ is said to be *prime* iff for all formulas α, β: α X or β X whenever $\alpha \quad \beta$ X.

Notice that a theory of $\vdash_J$ being maximal in the family of all theories to which a given fixed formula does not belong (so-called relatively maximal theory) is prime. This follows from the facts that $\vdash_J (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \quad \beta) \to \gamma))$ (the axiom from (HPL)), (MP) is a rule of $\vdash_J$ and that the deduction theorem (DT) holds for $\vdash_J$.

The crucial point for the completeness proof is the following:

2.1. Lemma. Let Π be the family of all prime theories of $\vdash_J$. For any X Π and any formula α: $\neg\alpha$ X iff Y Π (X Y (α Y

Y A)), where A = {Y Π: β (β,¬β Y)} (or equivalently A = {Y Π : {¬α: α L} Y}, L is the set of all formulas).

Proof. ( ): Obvious by definition of A.
( ): Assume that ¬α X. Since (α → ¬α) → ¬α X ((Ax2)) and X is closed on (MP): α → ¬α X. So X {α} $\nvdash_J$ ¬α by (DT). Due to Lindenbaum lemma let Y be a relatively maximal theory such that X Y, α Y and ¬α Y. Clearly Y Π. Moreover, Y A. In the case Y A, β,¬β Y for some β. However due to (Ax1): ¬β → (β → ¬α) Y. Hence ¬α Y. A contradiction. ■

Having the standard counterpart of Lemma 2.1 for the connectives , , → for any X Π, α, β L,

α β X iff α, β X,
α β X iff α X or β X,
α → β X iff Y Π (if X Y, then α Y or β Y),

one can define the canonical model for $\vdash_J$ of the form M = <Π, ,A,ν>, where for each p Var and X Π: X ν(p) iff p X, and next show inductively:

2.2. Lemma. For any X Π and any formula α: α is true in the canonical model at the point X iff α X.

Hence and from the fact that each relatively maximal theory of $\vdash_J$ is prime, the *completeness theorem*: $\models_{M(J)}$ $\vdash_J$ follows directly.

2.3. Remark. Kripke-style semantics for the minimal logic has been already presented, on the language with falsehood *f* instead of negation ¬. The characteristic feature of this semantics is that no truth condition for *f* is necessary (cf. [4] p. 40) except the one concerning preservation of truth: x,y W (x ≤ y (x $\models_M$ *f* y $\models_M$ *f*)). Introducing the definition ¬α = α → *f* we obtain in a model M of that semantics:
(¬ ‘) x $\models_M$ ¬α iff y W (x ≤ y (y $\models_M$ α y $\models_M$ *f*)).
Thus putting A = {x W: x $\models_M$ *f* }, (¬) immediately follows. So we obtain a model of our kind, however with A fulfilling the condition:
(A) x,y W (x ≤ y (x A y A)).

Going conversely, that is starting from our model M = <W,≤,A,ν> such that (A) holds true, and putting the definition $f = \neg(p \rightarrow p)$ we clearly obtain from (¬) that for any x W, x A iff $x \vDash_M f$, so we have (¬‘).

This shows that our semantics and the former do not coincide (up to translations ¬ ⇔ *f*); the condition (A) need not to be satisfied in our models. This however is quite inessential. At the very beginning one can restrict the models of our semantics to those satisfying the condition (A), yet it holds for the canonical model.

3. The strengthening of the Johansson's logic by the law of excluded middle

Now consider the axiomatic extension $\vdash_{JEM}$ of $\vdash_J$ by the axiom:

(EM) ¬α α.

To obtain $\vdash_{JEM}$ one can put simply (EM) instead of (Ax2) in the set of axioms for $\vdash_J$: (Ax2) follows by (DT) from (EM), (MP) and (HPL) (the above-mentioned axiom from (HPL) of the form: (¬α → ¬α) → ((α → ¬α) → ((¬α α) → ¬α))) .

It is easy to show that an adequate semantics for such a logic consists of all the models M = < W, ≤, A, ν > from M(J) with A satisfying the following condition:

(A1) x W (x A whenever x is not a minimal element of < W, ≤ >).

Notice that in such a model M, for any formula α and x W: $x \vDash_M \neg\alpha$ iff y W (x ≤ y ($y \vDash_M \alpha$ y A)) iff ($x \vDash_M \alpha$ x A) & y W (x ≤ y & x ≠ y ($y \vDash_M \alpha$ y A)). However, under (A1) the last conjunct always holds true, thus the truth condition for ¬ in M may be simplified to

(¬1) $x \vDash_M \neg\alpha$ iff ($x \vDash_M \alpha$ x A).

Having a straightforward soundness proof for $\vdash_{JEM}$, one may prove the completeness one, first proving a counterpart of Lemma 2.1:

3.1. Lemma. For any X   Π and any α: ¬α   X iff (α   X   X   A), Π is the family of all prime theories of $\vdash_{JEM}$, and A is defined as in 2.1;

and next showing that the canonical model < Π,   , A, ν > with A and ν defined as before, fulfils (A1).

Notice that the logic $\vdash_{JEM}$ is the subject of [7] where also an algebraic semantics for it is given. The Kripke-style semantics resembles the one presented here, however with general requirement of truth preservation for each formula by ordering of a model (the condition (**)) without (A1) explicitly formulated. Still, much earlier, Curry considered the logic as a system of so-called “strict negation” in various proof-theoretical modes under the names of LD, TD, HD ([3], pp. 260, 280, 285).

4. A general view on Kripke-style semantics

The generalization of Kripke semantics given below was received in result of unsuccessful attempts to get Kripke models for some weakenings of the Johansson’s minimal logic.

Let us call a *generalized Kripke model for a propositional language L* = (L, F_1,..., F_n) (for short gKm) simply a pair M = <W, $\vDash_M$> where W ≠   is any set and $\vDash_M$ W × L is any relation. $\vDash_M$ plays the role of forcing relation. For any class M of gKm’s, a consequence relation $\vDash_M$ P(L) × L defined as follows: X $\vDash_M$ α iff   M   M   x   W (x $\vDash_M$ α whenever   β   X, x $\vDash_M$ β) will be called the consequence determined by M.

We would prefer to call that notion “a general Kripke model”, however this term has been already used for a concept of smaller generality, in definition 7.6 of Corsi [1], where a Kripke model with a relation R   W × W is said to be general when some generalization of the condition of truth preservation for propositional variables (see (*), Sec. 2) implies a corresponding generalization of truth preservation for all the formulas (see (**)).

As one might expect, each consequence relation on the language L can be represented by some classes of gKm’s, that is, it coincides with some consequences determined by those classes:

4.1. Fact. Any consequence relation $\vdash$ on L, no matter if finitary or structural, is determined by a 1-element class composed of the gKm $\mathsf{M}(\vdash) = <\mathrm{Th}(\vdash), \models_{\mathsf{M}(\vdash)}>$, (i.e., $\vdash\ = \models_{\{\mathsf{M}(\vdash)\}}$), where $\mathrm{Th}(\vdash)$ is the family of all theories of $\vdash$ and $\models_{\mathsf{M}(\vdash)}$ is the converse of   , that is for X $\mathrm{Th}(\vdash)$, α L: X $\models_{\mathsf{M}(\vdash)} \alpha$ iff α X.

Proof. A simple calculation shows this result: for any X   L, α L, X $\models_{\{\mathsf{M}(\vdash)\}} \alpha$ iff   Y $\mathrm{Th}(\vdash)$(Y $\models_{\mathsf{M}(\vdash)} \alpha$ whenever for all β X, Y $\models_{\mathsf{M}(\vdash)} \beta$) iff   Y $\mathrm{Th}(\vdash)$(α Y whenever X   Y) iff α $\bigcap\{$Y $\mathrm{Th}(\vdash)$: X   Y$\}$ iff X $\vdash \alpha$. ■

4.2. Fact. Let $\vdash$ be a consequence relation defined on the language L by means of a set of rules R. Then $\vdash$ is determined by the class M(R) of all gKm's < W, $\models_{\mathsf{M}}$ > such that the forcing $\models_{\mathsf{M}}$ fulfils the condition: for any rule r   R, any sequent Y/ α r, and each x   W: x $\models_{\mathsf{M}} \alpha$ whenever β Y, x $\models_{\mathsf{M}} \beta$ (i.e., M(R) consists of all the gKm's M such that $\vdash$ $\models_{\{\mathsf{M}\}}$).

Proof. The inclusion $\vdash$ $\models_{\mathsf{M}(R)}$ holds due to definition of the class M(R). To show the converse inclusion notice that $<\mathrm{Th}(\vdash), \models_{\mathsf{M}(\vdash)}>$ M(R) (since every theory of $\vdash$ is closed on each rule from R). Therefore, $\models_{\mathsf{M}(R)}$ $\models_{\{\mathsf{M}(\vdash)\}} = \vdash$ accordingly Fact 4.1. ■

The parameters like relations and valuations of any ordinary Kripke model M always determine uniquely the "forcing" relation $\models_{\mathsf{M}}$. Therefore, any such M may be conceived as a generalized Kripke model. The converse, i.e., passing from a gKm to an "ordinary" Kripke model is in many cases not possible. The problem of existence of a Kripke semantics for a given propositional logic can be posed as a question on reproducing of less or more trivial class of gKm's to a class of Kripke models of a certain kind.

In the sequel three non-trivial gKm's semantics' for two weakenings of Johannson's logic are presented. There, contrary to the Kripke's usual approach, the forcing relations are essentially based on the non-ontological relation R   W × L being the core of the actual generalization.

5. A generalized Kripke semantics for two weakenings of Johansson's logic

Let $\vdash_1$ be a logic defined by (HPL), (MP) and (Ax1). We will show that the adequate semantics for $\vdash_1$ consists of all the gKm's $< W, \models_M >$, where $\models_M$ W × L is any relation fulfilling the truth conditions for propositional variables:

(*') p Var x,y W (x ≤ y ($x \models_M p$ $y \models_M p$)),

and for , , → from Section 2, and the following one for ¬:

(¬2) $x \models_M \neg\alpha$ iff y W (x ≤ y < y, α > R),

where R W × L is a relation satisfying the clause:

(R) x W (α (x $\models_M$ α & < x, α > R) β < x, β > R).

5.1. Remark. Now we are not interested in a relation $\models_M$ being uniquely inductively defined by the same parameters ≤, ν (without A W) with the same truth conditions for propositional variables and for , , → as in Section 2, and by the additional parameter R W × L fulfilling (R) with the truth condition (¬2). Albeit such an induction sometimes goes through dependently on the relation R (consider for example a trivial case R = and instead of (*') introduce a valuation ν with (*) and the truth condition for propositional variables from Section 2). We only stipulate the three set-theoretical objects: ≤, R, $\models_M$, the last one being the most important to define a consequence relation, and six conditions satisfied by them.

Now the soundness proof follows immediately since the condition (**) from Section 2 holds here. To show the validity of (Ax1), we need the clauses (**), (¬2) and (R).

In order to show the completeness, consider the canonical gKm M = < Π, $\models_M$ >, where Π is the family of all prime theories of $\vdash_1$, and $\models_M$ is defined similarly as in Fact 4.1: X $\models_M$ α iff α X. The relation $\models_M$ is correlated with the following two parameters: inclusion as an ordering, and a relation R Π × L: < X, α > R iff ¬α X. Then the condition (R) now of the form: X Π (α (α X & < X, α > R) β < X, β > R) is clearly satisfied due to (Ax1) and (MP).

The expression: for any X Π, $\neg\alpha$ X iff Y Π (X Y $\neg\alpha$ Y) evidently holds true, so by definition of R, ($\neg$2) is satisfied. The condition (*') clearly holds and the same is for the remaining ones as in the case of Hilbert's positive logic (cf. the conditions just before Lemma 2.2). From this the completeness theorem follows directly.

Any model M = < W, ≤, A, ν > for the logic $\vdash_J$, rewritten in the form < W, $\models_M$ >, where $\models_M$ is defined inductively as in Section 2, is a gKm for $\vdash_1$: for each x W and α L put < x, α > R iff x $\not\models_M$ α or x A.

The axiom (Ax2) is not valid here. Consider for example any gKm < W, $\models_M$ > for $\vdash_1$, where $\models_M$ is defined by R = . Then, the condition ($\neg$2) reduces to: ($\neg$2') x $\not\models_M$ $\neg\alpha$. The relation $\models_M$ can be inductively defined by a valuation ν: Var $\rightarrow$ P(W) fulfilling (*), the truth conditions for variables, for , , $\rightarrow$ from Section 2, and ($\neg$2'). So put ν(p) = . Then we have in the model: x $\not\models_M$ (p $\rightarrow$ $\neg$p) $\rightarrow$ $\neg$p, for any x W.

Now, let us consider the logic $\vdash_2$ defined by (HPL), (MP) and (Ax2).

We will show two generalized Kripke semantics' for the logic. The first consists of all the gKm's < W, $\models_M$ > where $\models_M$ is correlated with the parameters: ≤ (reflexive and transitive on W) and R W × L, and fulfils the same conditions as in the semantics for logic $\vdash_1$, except ($\neg$2) and (R). The truth condition for $\neg$ is of the form:

($\neg$3) x $\models_M$ $\neg\alpha$ iff y W (x ≤ y (y $\models_M$ α < y, α > R)).

where R W × L is any relation.

The *soundness theorem* for $\vdash_2$ clearly holds. In order to show the completeness let us take into account the following

5.2. Lemma. X Π ($\neg\alpha$ X iff Y Π (X Y (α Y $\neg\alpha$ Y))), α is any formula and Π is the family of all prime theories of $\vdash_2$.

Proof. ( ): Obvious.

( ): Let $\neg\alpha$ X. Then make use of (Ax2), (MP) and (DT) as in the proof of Lemma 2.1 to the effect that X {α} $\nvdash_2$ $\neg\alpha$. Next apply the

Lindenbaum lemma to obtain a Y Π such that X Y, α Y and $\neg\alpha$ Y. ■

Now, consider the canonical gKm $M = \langle \Pi, \vDash_M \rangle$ where for any X Π, α L: X $\vDash_M$ α iff α X and the parameters: , and R: $\langle X, \alpha \rangle$ R iff $\neg\alpha$ X. Then, according to Lemma 5.2, the condition ($\neg$3) holds in M, so the *completeness theorem* follows.

The second semantics differs from the first only by the condition:

($\neg$3') x $\vDash_M \neg\alpha$ iff y W (x $\leq$ y (y $\vDash_M \alpha$ y $\vDash_M \neg\alpha$))

put instead of ($\neg$3). In that way the relation $\vDash_M$ in a gKm $\langle W, \vDash_M \rangle$ is now independent of a relation R W × L.

It is obvious, due to Lemma 5.2, that the second semantics is adequate for the logic.

Equivalently the class of gKm's for $\vdash_2$ may be defined as consisting of all the models $\langle W, \vDash_M \rangle$, where $\vDash_M$ fulfils (**) and the conditions for , , $\rightarrow$ from Section 2, and

($\neg$3'') x $\vDash_M \neg\alpha$ whenever y W (x $\leq$ y (y $\vDash_M \alpha$ y $\vDash_M \neg\alpha$)) .

Any model $\langle W, \leq, A, v \rangle$ for the minimal Johansson's logic determines inductively the relation $\vDash_M$ W × L such that $\langle W, \vDash_M \rangle$ is a gKm belonging to the second semantics for $\vdash_2$: it is enough to show that $\vDash_M$ fulfils the condition ($\neg$3''). So suppose that ($\neg$3'') is not satisfied by $\vDash_M$. Then for an x W: (1) x $\nvDash_M \neg\alpha$ and (2) y W (x $\leq$ y (y $\vDash_M \alpha$ y $\vDash_M \neg\alpha$)). From (1) and ($\neg$) it follows that for some y W, x $\leq$ y and (3) y $\vDash_M \alpha$, (4) y A. From (2) and (3) we have: y $\vDash_M \neg\alpha$ so again from ($\neg$), reflexivity of $\leq$ and (3) we obtain that y A. A contradiction with (4).

Similarly, such a $\langle W, \vDash_M \rangle$ is a gKm belonging to the first semantics for $\vdash_2$: define only the relation R as follows: $\langle x, \alpha \rangle$ R iff x A.

In fact, those two semantics for $\vdash_2$ coincide: any model $\langle W, \vDash_M \rangle$ where $\vDash_M$ depends on a relation R, and fulfils ($\neg$3) is a gKm such that $\vDash_M$ fulfils ($\neg$3'): we should show only ( )-part of ($\neg$3'), (since the ( ) -part is true due to (**)), so the implication: if y W

$(x \leq y \quad (y \models_M \alpha \quad y \models_M \neg\alpha))$, then $y \quad W\ (x \leq y \quad (y \models_M \alpha$ $< y, \alpha > \quad R))$, which is trivial due $(\neg 3)$ and reflexivity of $\leq$. Conversely, in any gKm $<W, \models_M>$ from the second semantics, where $\models_M$ fulfils $(\neg 3')$, it suffices to define a relation R as $< x, \alpha > \quad R$ iff $x \models_M \neg\alpha$, to obtain a gKm from the first semantics.

Notice that the axiom (Ax1) is not valid in the semantics. Consider for example such a model $< W, \models_M >$ of the first kind, where $\models_M$ is defined by the following $R \quad W \times L$: $< x, \alpha > \quad R$ if $Var(\alpha) = \{p\}$, where p is any fixed variable, $Var(\alpha)$ is the set of all propositional variables occurring in α. Furthermore, $\leq = id_W$ (the identity on W) and $x \models_M p$, $x \models_M q$ for some $x \quad W$ and $q \quad Var$ such that $q \neq p$. Clearly then we have $x \models_M \neg p$ and $x \not\models_M \neg q$ that is $x \not\models_M \neg p \to (p \to \neg q)$.

6. The Johansson's logic with classical implication

Let us consider the Johansson's minimal negation based on classical positive logic, that is the extension $\vdash_{ClJ}$ of the minimal logic $\vdash_J$ by the axiom:

$(Cl\to) \quad \alpha \quad (\alpha \to \beta)$

Notice that $\vdash_{ClJ} \neg\alpha \quad \alpha$. The proof is as follows:
$(\alpha \to \neg\alpha) \to \neg\alpha$ (Ax2), $\neg\alpha \to (\neg\alpha \quad \alpha)$ (HPL), $(\alpha \to \neg\alpha) \to (\neg\alpha \quad \alpha)$ (HPL)+(MP), $\alpha \quad (\alpha \to \neg\alpha)$ $(Cl\to)$, $\alpha \to (\neg\alpha \quad \alpha)$ (HPL), $(\alpha \to (\neg\alpha \quad \alpha)) \to (((\alpha \to \neg\alpha) \to (\neg\alpha \quad \alpha)) \to ((\alpha \quad (\alpha \to \neg\alpha)) \to (\neg\alpha \quad \alpha)))$ (HPL), $\neg\alpha \quad \alpha$ (MP).

It is not difficult to show that an adequate semantics for the logic consists of all the models for the minimal logic in which the relation $\leq$ is the identity on W. In such models the truth conditions for $\to$ and $\neg$ are transformed into the following ones:

$(\to) \quad x \models_M \alpha \to \beta \quad$ iff $\quad (x \models_M \alpha \quad x \models_M \beta)$,
$(\neg 1) \quad x \models_M \neg\alpha \quad$ iff $\quad (x \models_M \alpha \quad x \quad A)$.

The completeness theorem for $\vdash_{ClJ}$ clearly follows from the following

6.1. Lemma. For any prime theory X of $\vdash_{ClJ}$ and any formulas α,β:
(1) $\alpha \to \beta \quad X$ iff $(\alpha \quad X$ or $\beta \quad X)$,

(2) $\neg\alpha \quad X$ iff $(\alpha \quad X \quad X \quad A)$, where A is defined as in Lemma 2.1.

Proof. (1) ( ): by (MP).
(1)( ): by (HPL) and (Cl→).
(2)( ): by definition of A.
(2)( ): suppose that $\neg\alpha \quad X$, and that $\alpha \quad X$ implies $X \quad A$. Then by (Ax2) and (MP): $\alpha \to \neg\alpha \quad X$, consequently from (Cl→): $\alpha \quad X$ (or immediately from (EM)). Thus $X \quad A$, hence for some β: $\beta, \neg\beta \quad X$ and by (Ax1) and (MP): $\neg\alpha \quad X$. A contradiction. ■

As in the case of $\vdash_{JEM}$, the logic $\vdash_{ClJ}$ has been already considered in [3, pp. 260, 280, 286] in different proof-theoretical forms as the systems LE, TE, HE. Moreover, [2] is devoted to that logic, where it appears under the name of $\vdash_{CAR}$; [2] contains its valuational semantics as well as a proof of decidability. Compare also [11]. The note [8] shows a matrix semantics for the logic and some interesting property: the set of formulas $\{\alpha: \vdash_{ClJ} \alpha\}$ is a maximal system among all the (invariant) systems closed on (MP) and properly included in the set of all classical tautologies. As far as we know, among the well-known paraconsistent systems only the system P^1 of A. M. Sette [9] has that property.

7. The logics $\vdash_1, \vdash_2$ with classical implication

Now consider the logic $\vdash_{Cl1}$ being the extension of $\vdash_1$ by the axiom (Cl→).

As one could expect an adequate semantics for $\vdash_{Cl1}$ consists of all the gKm's $< W, \models_M >$ for $\vdash_1$ such that the relation $\models_M$ is correlated with the ordering $\leq = id_W$. Therefore the truth condition for $\neg$ is now of the form: $x \models_M \neg\alpha$ iff $< x, \alpha > \quad R$, where R satisfies the condition (R). In that way we can define a gKm $<W, \models_M>$ for the logic without the parameter R, as follows. The relation $\models_M$ is to fulfil the following conditions:

$x \models_M \alpha \quad \beta$ iff $x \models_M \alpha$ & $x \models_M \beta$,
$x \models_M \alpha \quad \beta$ iff $x \models_M \alpha$ or $x \models_M \beta$,
$x \models_M \alpha \to \beta$ iff $(x \models_M \alpha \quad x \models_M \beta)$,
$\alpha (x \models_M \alpha$ & $x \models_M \neg\alpha) \quad \beta\, (x \models_M \neg\beta)$,

where the last one can be regarded as trivial due to the validity of (Ax1) and the classical interpretation of $\rightarrow$.

It is easily seen that the relation $\models_M$ from a Kripke model for $\vdash_{ClJ}$ satisfies also all the above truth conditions, i.e., each Kripke model for $\vdash_{ClJ}$ is a gKm for $\vdash_{Cl1}$.

Clearly the completeness theorem follows from

7.1. Lemma. For any prime theory X of $\vdash_{Cl1}$: if $\alpha(\alpha \quad X \ \& \ \neg\alpha \quad X)$, then $\beta(\neg\beta \quad X)$.

Proof. Obvious due to (Ax1) and (MP). ■

Now consider the extension $\vdash_{Cl2}$ of $\vdash_2$ by (Cl$\rightarrow$). Analogously as before, the gKm's $< W, \models_M >$ for $\vdash_{Cl2}$ are those among gKm's for $\vdash_2$ (of the first or second kind) that the relation $\models_M$ is correlated with the ordering $\leq \, = id_W$. Thus the truth condition for $\neg$ is now of the form:

$x \models_M \neg\alpha$ iff $(x \models_M \alpha \qquad <x, \alpha> \quad R)$ in the models of the first kind, and

$x \models_M \neg\alpha$ iff $(x \models_M \alpha \qquad x \models_M \neg\alpha)$ in the models of the second kind.

The last condition for $\neg$ is equivalent to the following:

$x \not\models_M \alpha \qquad x \models_M \neg\alpha$.

Similarly to the case of $\vdash_{Cl1}$, each Kripke model for $\vdash_{ClJ}$ is a gKm for $\vdash_{Cl2}$.

7.2. Lemma. For any prime theory X of $\vdash_{Cl2}$ and any formula α: $\neg\alpha \quad X$ whenever $\alpha \quad X$.

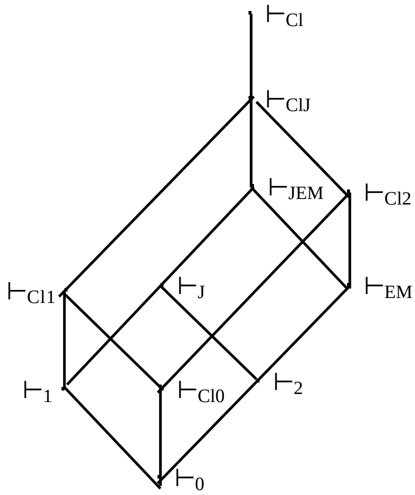
Proof. Obvious due to (Ax2), (Cl$\rightarrow$) and (MP). ■

On the base of Lemma 7.2 the completeness proof follows.

The logic $\vdash_{Cl2}$ also has been already considered: it appears under the name of <1> in [10] with the axiomatization containing (EM) instead of (Ax2). Clearly (EM) is provable in $\vdash_{Cl2}$ (the same argument as for $\vdash_{ClJ}$ in Sec. 6), conversely, (EM) yields (Ax2) (cf. Sec. 3).

8. Appendix

The logics being the subject of the paper form the following lattice, where $\vdash_0$ and $\vdash_{Cl0}$ are the Hilbert's and classical positive logic respectively, on the language with negation, $\vdash_{Cl}$ is the classical logic, and $\vdash_{EM}$ is the extension of $\vdash_2$ by (EM), that is, simply the extension of $\vdash_0$ by (EM).



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