Logica Trianguli, 1, 1997, 15-31

GENERALISED SEQUENT CALCULUS FOR PROPOSITIONAL MODAL LOGICS

Andrzej INDRZEJCZAK

Abstract

The paper contains an exposition of some non standard approach to gentzenization of modal logics. The first section is devoted to short discussion of desirable properties of Gentzen systems and the short review of various sequential systems for modal logics. Two non standard, cut-free sequent systems are then presented, both based on the idea of using special modal sequents, in addition to usual ones. First of them, **GSC I** is well suited for non-symmetric modal logics (accessibility relation in their Kripke models is not symmetric) The second one, **GSC II** is devised specially for symmetric, i.e. **B**-logics. **GSC I** and **GSC II** are not different formalizations, from the theoretical point of view **GSC I** may be seen as a simplification of the more general approach present in **GSC II**. They are considered separately, mainly because **B**-logics demand different, and more complicated, strategy in completeness proof, whereas non symmetric logics are easily and uniformly characterised by means of Fitting's *Consistency Properties*. The weakest modal logic captured by this formalization is minimal regular logic **C**, but many stronger logics are obtainable by addition of suitable *structural* rules, which conforms to Dośen's methodology. Both variants of **GSC** satisfy also other, besides cut-freedom, desirable properties.

1. Introduction

The story of gentzenization of modal logics is quite long; certainly older than fruitful semantical investigations due to Kripke, Hintikka and others. Feys [50] and Curry [52] seems to be the first, but many successors follow. Nevertheless, Bull & Segerberg [84] claimed that only exceptional modal logics are characterizable in terms of simple and natural rules. This opinion was probably the effect of dissatisfaction with resulted formalisations. Although some modal logics had obtained suitable sequential form by then, there were still many without. Moreover, no general approach was ready at the moment. According to Sambin & Valentini [82] the problem is not to find suitable sequential rules for some modal logic, but to find such rules that satisfy some special properties, usually connected with sequent systems. In other words, the problem is not to construct a sequent system, but to construct a *good* sequent system.

What is then a good sequent system? Properties that attract logicians, when comparing sequent systems with axiomatic ones, are usually modelled on the original Gentzen rules for classical logic. Following Curry [63], Wansing [92] and Avron [91], we make a short list of the most important These are the features of the rules, in the first instance, but can be predicted also on the whole system, terminology is not established rigidly by now, so one can find these properties under different names in other papers.

1) *Subformula-property*: each wff displayed in premise-sequent(s) is present as a subformula (proper or not) in conclusion. Elimination of Cut-rule is necessary (and sometimes even sufficient) condition for the system to have this property, hence often the term *cut-freedom* is applied interchangeably. But it should be remembered that many cut-free sequent systems do not have subformula-property.
2) *Invertibility*: not only conclusion-sequent follows from premise-sequent(s), but also premise-sequent(s) from the conclusion, i.e. rules are doubly sound.
3) *Separation*: a rule for logical constant @ does not exhibit any other constants in premise(s) and conclusion.
4) *Symmetry*: each constant has a pair of rules for introducing it into an antecedent and a succedent of a conclusion-sequent.
5) *Explicitness*: rules for constants exhibit them only in conclusion-sequents, never in premises; moreover they should exhibit only one occurrence of the constant.
6) *Purity*: soundness of a rule is not affected by addition of any wffs both to conclusion-sequent and to all premise-sequents (either in antecedent or in succedent-position).
7) *Interdefinability*: this is special property for modal logics; both $\Box$ and $\Diamond$ should have independent rules, which allows to prove all definitional equivalences between them.
8) *Structurality (Dośen principle)*: for any family of logics in the same language: rules for logical constants are constant, different systems are characterizable exclusively in terms of structural rules.

This list is not exhaustive. But our aim is not to discuss here all important properties, noticed since thirties. Neither do we try to describe the implications of having (or lacking) some of them for decidability and other questions. One may consult, e.g. aforementioned authors. But it should be noted that many well known sequent systems are defective in some way with respect to this list. Before we make a few observations it would be nice to divide existing systems. Let us call a system *standard* if it is conservative with respect to original Gentzen system and only new rules for modal constants are added; otherwise it is *non standard*. Among the first group there are systems of Ohnishi & Matsumoto [57], Kanger [57] (except his system for **S5**), Zeman [73] to mention just a few authors. Systems of this sort have many virtues; rules are simple and self-evident and some of the most popular modal logics, including **K, T, S4, GL**, obtained, practically simple proof procedures. But beside subformula-property and separativity all other postulates are broken, what is more, in the case of **S5** even subformula-property fails.

Among non standard systems one may naturally distinguish two groups: there are formalizations devised specially for some more important logics, especially for **S5**, and some which are examples of global modifications of Gentzen format, made not necessarily with modal logics in mind. Kanger's [57] system for **S5** is probably the first serious modification, where basic items in sequents are not wffs but *labelled* wffs; this approach was further developed and generalized on other logics by Fitting [83] (but for a kind of Beth tableaux) and Wansing [95]. Also the systems of Mints [70] and Sato [80] are rather of local character; changes in them are based on special properties of **S5**. Both systems are cut-free but subformula-property fails.

Many non standard sequent systems introduce additional operations on wffs or their sets, and additional structural rules. Such substantial enrichment allows for greater flexibility in modelling various logics. From the viewpoint of modal logics the most important proposal is *Display Logic* of Belnap [82], (simplified with respect to modal logics only, by Wansing [92]). Trzęsicki [84] (for temporal logics) and Cerrato [90] represent similar kind of deviation from standard Gentzen format. One of the most radical generalizations is that of Dośen [85], where sequents of higher levels are involved.

There is no place for thorough discussion of the advantages (and disadvantages) of all these (and many others) formalizations. More comments one may find in Wansing [92]. One thing should be noted, however; it seems that some of them are highly complicated in practice, because of richness of formal apparatus. In contrast, the present system, called **GSC** (Generalized Sequent Calculus), was constructed mainly with one purpose in mind - *simplicity* of proofs. Of course simplicity is a vague notion and hardly subject to any objective criteria, nevertheless we can at least make some introductory explication. Here, two criteria of simplicity were borne in mind; first, the rules for non modal constants should remain as close as possible to original Gentzen rules. Second, the number of additional elements should be kept within reasonable bounds. Main modifications involve the introduction of non classical sequents and one operation on wffs. The idea of using non classical sequents in cooperation with modal constants goes back to Curry [52] and Zeman [73]; here, it is the main instrument of dealing with modality in an uniform fashion.

2. General Presentation of GSC I

Language of the system is standard but with some additional technical devices: We list here basic symbols and conventions:

ϕ, ψ, possibly with subscripts, denote wffs, Γ, Δ, Σ, Π, denote finite sets of wffs. There are three types of sequents: one *classical* $\Gamma \quad \Delta$ and two *modal* ones,

$\Gamma \Box \quad \Delta$ and $\Gamma \Diamond \quad \Delta$; in the description of rules we use the convention Γ ( ) Δ to denote any sequent, whenever the type is irrelevant.

Any wff ϕ of standard propositional modal language (a PML-formula for short) may be prefixed with '-' in the course of proof. This sign is not iterated! it is only a sign of *shifting* a wff from one side of a sequent to the other. Any PML-formula so prefixed will be called *S-formula* and S-formulae are not combined according to usual rules of formation, thus such things as --ϕ, ϕ -ψ are not wffs at all. The set of wffs is simply the union of PML- and S-formulae. We also use the following convention: for any PML-formula ϕ let ϕ^* denote -ϕ and (-ϕ)* denote ϕ. Accordingly Γ^* denote $\{\phi^*: \phi \quad \Gamma\}$. This operation of shifting is necessary in **GSC,** because we have to replace original Gentzen rules by their *symmetric* variants, where the application of no logical rule shifts some wff from one side of sequent to another. This is a consequence of using modal sequents, where shifting of wffs is limited, hence some of the standard logical rules (namely for $\rightarrow$ and $\neg$) would be of no use. Separation of the process of introducing constants from that of shifting wffs allows to apply all rules for classical constants on any sequents, independently of their type.

In what follows, we call *A-formula* any wff, where the main connective is classical, *B-formula* any $\Box\phi$ or -$\Diamond\phi$ in the antecedent of a sequent, and $\Diamond\phi$ or -$\Box\phi$ in the succedent, *C-formulae* are defined dually with respect to the position in a sequent. Accordingly BΓ (CΓ) denotes a set of B (C) -formulae. VAR is the set of all propositional variables and AT = VAR VAR*. Sometimes, we also use the convention of writing ¬Γ, □Γ, ◇Γ in place of {¬φ : φ Γ}, {□φ : φ Γ}, {◇φ : φ Γ}, respectively, and Γ, Γ as a shortcut for conjunction (disjunction) of all wffs from the finite set Γ.

We divide the rules of the system into 4 groups:

1) Structural rules (general):

AX) $\phi \quad \phi$

W) $\dfrac{\Gamma\ (\quad)\ \Delta}{\phi\ \Gamma\ (\quad)\ \Delta}$ W) $\dfrac{\Gamma\ (\quad)\ \Delta}{\Gamma\ (\quad)\ \Delta\ \phi}$

2) Shifting rules:

*) $\dfrac{\phi\ \Gamma \quad \Delta}{\Gamma \quad \Delta\ \phi^*}$ *) $\dfrac{\Gamma \quad \Delta\ \phi}{\phi^*\ \Gamma \quad \Delta}$ TR.) $\dfrac{\Gamma\ \Box \quad \Delta}{\Delta^*\ \Diamond \quad \Gamma^*}$

*◇) $\dfrac{\text{-}\phi\ \Gamma\ \Diamond \quad \Delta}{\Gamma\ \Diamond \quad \Delta\ \Diamond\phi}$ *□) $\dfrac{\Gamma\ \Box \quad \Delta\ \text{-}\phi}{\Box\phi\ \Gamma\ \Box \quad \Delta}$ TR.) $\dfrac{\Gamma\ \Diamond \quad \Delta}{\Delta^*\ \Box \quad \Gamma^*}$

3) Logical rules:

¬) $\dfrac{\text{-}\phi\ \Gamma\ (\quad)\ \Delta}{\neg\phi\ \Gamma\ (\quad)\ \Delta}$ ¬) $\dfrac{\Gamma\ (\quad)\ \Delta\ \text{-}\phi}{\Gamma\ (\quad)\ \Delta\ \neg\phi}$

) $\dfrac{\phi\ \psi\ \Gamma\ (\quad)\ \Delta}{\phi \quad \psi\ \Gamma\ (\quad)\ \Delta}$) $\dfrac{\Gamma\ (\quad)\ \Delta\ \phi \qquad \Pi\ (\quad)\ \Sigma\ \psi}{\Gamma\ \Pi\ (\quad)\ \Delta\ \Sigma\ \phi \quad \psi}$

) $\dfrac{\phi\ \Gamma\ (\quad)\ \Delta \qquad \psi\ \Pi\ (\quad)\ \Sigma}{\phi \quad \psi\ \Gamma\ \Pi\ (\quad)\ \Delta\ \Sigma}$) $\dfrac{\Gamma\ (\quad)\ \Delta\ \phi\ \psi}{\Gamma\ (\quad)\ \Delta\ \phi \quad \psi}$

→) $\dfrac{\text{-}\phi\ \Gamma\ (\quad)\ \Delta \qquad \psi\ \Pi\ (\quad)\ \Sigma}{\phi\to\psi\ \Gamma\ \Pi\ (\quad)\ \Delta\ \Sigma}$ →) $\dfrac{\Gamma\ (\quad)\ \Delta\ \text{-}\phi\ \psi}{\Gamma\ (\quad)\ \Delta\ \phi\to\psi}$

□) $\dfrac{\phi \quad \Delta}{\Box\phi\ \Box \quad \Delta}$ □) $\dfrac{\Gamma\ \Box \quad \phi}{\Gamma \quad \Box\phi}$

◇) $$\frac{\phi\ \Diamond\quad \Delta}{\Diamond\phi\quad \Delta}$$

◇) $$\frac{\Gamma\quad \phi}{\Gamma\ \Diamond\quad \Diamond\phi}$$

4) Special structural rules:

K) $$\frac{\Gamma}{\Gamma\ \Diamond}$$

K) $$\frac{\Delta}{\Box\quad \Delta}$$

D) $$\frac{\Gamma\ \Box}{\Gamma}$$

D) $$\frac{\Diamond\quad \Delta}{\Delta}$$

T) $$\frac{\Gamma\ \Box\quad \Delta}{\Gamma\quad \Delta}$$

T) $$\frac{\Gamma\ \Diamond\quad \Delta}{\Gamma\quad \Delta}$$

4) $$\frac{B\Gamma\quad \Delta}{B\Gamma\ \Box\quad \Delta}$$

4) $$\frac{\Gamma\quad B\Delta}{\Gamma\ \Diamond\quad B\Delta}$$

Let us call the calculus given by these rules **GSC-I**. Rules from group 1-3 constitute modal logic **C**, the weakest *regular logic*, axiomatically characterised by one axiom and one rule over Classical Propositional Calculus:

P-N) $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$ RR) if $\vdash\ \Gamma\rightarrow \phi$ then $\vdash\ \Box\Gamma\rightarrow \Box\phi$

(see, e.g. Chellas [80], where it is called **R**, Fitting [83] or Perzanowski [89]).

Group 4 contains special rules that in different combinations characterise many of the most popular modal logics, in particular K) gives the weakest *normal modal logic* **K**, where RR) is replaced by Gödel's Rule:

GR) if $\vdash \phi$ then $\vdash \Box\phi$

Rules D), T) and 4) allow for strengthenings of **C** or **K** usually obtained by addition of suitable axioms:

D) $\Box\phi \rightarrow \Diamond\phi$ T) $\Box\phi \rightarrow \phi$ 4) $\Box\phi \rightarrow \Box\Box\phi$

GSC-*proof* of Γ Δ is a binary tree, rooted in this sequent, where each branch starts with axiom and runs down in accordance with the rules. It will be noted, shortly **GSC** ⊢ Γ Δ .

It is important to note that only ☐), ☐), ◇), ◇) and some of the structural rules change classical sequent into modal and vice versa; all other rules are conservative with respect to the sequent type of premise(s); in the case of two-premises rules both premises must be of the same type.

In contrast to standard Gentzen systems for modal logics our rules satisfy most of the properties from the previous paragraph. **GCS** is cut-free and has a *Generalised Subformula Property*, defined as follows:

For any PML-formula ϕ, If ϕ or -ϕ is used in the course of proof of Γ Δ, then ϕ SubFor(Γ Δ).

Moreover, all considered extensions of **C** are determined only by structural rules. Interested reader may easily check for himself if all other properties defined in the preceding paragraph apply to **GSC**-rules. Below there is an example of **S4-GSC I**-proof:

AX)	p p		
☐)	☐p ☐ p	☐q ☐q	AX)
TR)	-p ◇ -☐p	☐☐q ☐ ☐q	☐)
¬)	¬ p◇ -☐p	☐☐q ☐q	T)
◇)	◇¬ p -☐p	☐☐q -☐q	*)
*)	☐p ◇¬ p	☐☐q ¬ ☐q	¬)
)	☐p ☐☐q ◇¬ p ¬☐q		
*)	☐p ☐☐q -(◇¬ p ¬☐q)		
4)	☐p ☐☐q ☐ -(◇¬ p ¬☐q)		
¬)	☐p ☐☐q ☐ ¬ (◇¬ p ¬☐q)		
)	☐p ☐☐q ☐ ¬ (◇¬ p ¬☐q)		
☐)	☐p ☐☐q ☐¬ (◇¬ p ¬☐q)		
*)	-(☐p ☐☐q) ☐¬ (◇¬ p ¬☐q)		
→)	(☐p ☐☐q) → ☐¬ (◇¬ p ¬☐q)		

In order to prove *soundness* of our systems we consider the following translations τ and ℑ from the set of wffs and sequents of **GSC** into the set of PML-formulae:

for any PML-formula ϕ:
$\tau(\text{-}\phi) = \neg\phi$
$\tau(\phi) = \phi$
$\Im(\Gamma \quad \Delta) = \quad \tau(\Gamma\ ) \rightarrow \quad \tau(\Delta)$
$\Im(\Gamma \,\Box \quad \Delta) = \quad \tau(\Gamma\ ) \rightarrow \Box(\ \tau(\Delta))$
$\Im(\Gamma \,\Diamond \quad \Delta) = \Diamond(\ \tau(\Gamma)) \rightarrow \quad \tau(\Delta)$

In case of an empty antecedent we consequently put 0-ary constant $\top$ for Γ and for an empty succedent we put 0-ary constant . It is independent of the type of sequent, e.g. sequent $\Gamma \,\Box$ is replaced by a formula $\Gamma \rightarrow \Box$. This translation enables us to prove the soundness of **GSC**

Theorem 1. If **GSC-L** $\vdash \Gamma \quad \Delta$ then **L** $\models \Im(\Gamma \quad \Delta)$, where **L** is one of the extension of **C**, obtained by combination of axioms D), T), 4), and possibly, adding of RG).

Proof. By induction on the length of proof. It is sufficient to check that each **GSC**- rule is **C**-sound (sound in *augmented Kripke frames*, see e.g. Fitting [83]) under the translation $\Im$. Also special structural rules are sound in the suitable classes of frames.

We may strengthen a bit our last result, which will be of some importance in the next section. Except Weakening (W) and W)), *$\Diamond$I), *$\Box$), $\Box$) and $\Diamond$) all other rules in group 1) -3) are *doubly sound*, hence not only premise(s) yield conclusion but also conclusion entails (both) premise(s).

Usually Cut is considered in Gentzen calculi as one of the primitive rules, then proof of its eliminability is provided for the sake of effective decision procedure. This strategy is very convenient because the proof of *completeness* is straightforward in the presence of Cut; we simply present proofs of suitable axioms. Lack of Cut forces to construction of some algorithm that enables either to get a proof or, if there is none, to provide a countermodel. This will be the strategy we follow in the next section when faced with **B**-logics. For the time being we can do without any effective procedure, basing on the results of Fitting [73]. Smullyan [68] showed for the classical logic that usual Lindenbaum construction in completeness proof may be essentially weakened in the way that no use of Modus Ponens (hence Cut) is necessary. Fitting [73] and

[83] contain an extension of this strategy to many modal logics, including those we have considered. The method, in a nut-shell, is that for the logic in question we define the set of *Consistency Properties* which are sufficient to build a model. Any formalization of this logic is then established to be complete, if we can show that consequence relation defined by this calculus satisfies these properties. The details of model construction are to be found in Fitting [83], we limit ourselves to the presentation of the essential (modal) conditions and proof that our calculi satisfy them. Actually, in Fitting [83], these conditions are given in a compact way by means of some conventions we do not use, hence we must slightly modify them.

Any family of sets of wffs is **CP-L** (the set of Consistency Properties with respect to the logic **L**) if for each Γ **CP-L** :

a) Γ satisfies suitable conditions for propositional constants
b) if (Γ is normal and) ◇ϕ Γ then Γ# {ϕ} **CP-L**
c) if (Γ is normal and) ¬☐ϕ Γ then Γ# {¬ϕ } **CP-L**
d) if (Γ is normal and) Γ **CP-L** then Γ# **CP-L**
e) if ☐ϕ Γ then Γ {ϕ} **CP-L**
f) if ¬◇ϕ Γ then Γ {¬ϕ } **CP-L**

Γ is *normal* iff at least one ☐ϕ (or ¬◇ϕ) belongs to Γ; this condition is added in the case of non normal logics. Conditions a), b), c) have to be satisfied by all logics, condition d) by all logics with *serial accessibility relation* in Kripke models, e), f) by all *reflexive* logics. Γ# is defined as {ϕ:☐ϕ Γ} {¬ϕ :¬◇ϕ Γ}, for all *non-transitive* logics. In the case of *transitive* and *reflexive* logics it is simply {☐ϕ:☐ϕ Γ} {¬◇ϕ:¬◇ϕ Γ}, in the case of *transitive* but not *reflexive* logics (like **K4**) it is the union of Γ# for both *transitive* and *non-transitive* logics.

Now let us define a family of finite sets of wffs as follows. Put Γ ¬Δ in it if the sequent Γ Δ is ***not-provable*** in **GSC-I.** It is easily verifiable that so defined family of sets of wffs is **CP-L**. For the sake of illustration we check the condition b) for **GSC-K**.

Assume that, ◇ϕ is in some Γ ¬Δ but (Γ ¬Δ)# {ϕ} **CP-K**, hence ϕ Γ# Δ# is **GSC-K-**provable, and we can supply a following proof of Γ Δ on this basis:

given	$\phi\ \Gamma\#\quad \Delta\#$
	↓
*) n-times	$\phi\ \Gamma\#\ \text{-}(\Delta\#)$
K)	$\phi\ \Gamma\#\ \text{-}(\Delta\#)\ \Diamond$
	↓
*◇) n-times	$\phi\ \Gamma\#\ \Diamond\quad \Diamond(\Delta\#)$
TR.)	$\text{-}(\Diamond(\Delta\#))\Box\quad \text{-}\phi\ \text{-}(\Gamma\#)$
	↓
*☐) m-times	$\text{-}(\Diamond(\Delta\#))\ \Box(\Gamma\#)\Box\quad \text{-}\phi$
TR.)	$\phi\ \Diamond\quad \Diamond(\Delta\#)\ \text{-}(\Box(\Gamma\#))$
◇)	$\Diamond\phi\quad \Diamond(\Delta\#)\ \text{-}(\Box(\Gamma\#))$
	↓
*) m-times	$\Diamond\phi\ \Box(\Gamma\#)\quad \Diamond(\Delta\#)$
	↓
W) and W) k-times	$\Gamma\quad \Delta$

where n, m is the cardinality of $\Delta\#$ and $\Gamma\#$, respectively, and k is the cardinality of $\Gamma\quad \Delta$ -($\Gamma\#\quad \Delta\#\quad \{\Diamond\phi\}$).

Thus, for any sequent, if $\Gamma\quad \Delta$ is not provable in **GSC-L** , then $\Gamma\quad \neg\Delta$ **CP-L**, and there is a suitable **L**-Kripke model which satisfies $\Gamma\quad \neg\Delta$, hence $\Gamma\quad \Delta$ is **L**-non valid. It is stated as

Theorem 2. If **L** $\models \Im(\Gamma\quad \Delta)$ then **GSC-L** $\vdash \Gamma\quad \Delta$, where **L** is one of the extension of **C**, obtained by combination of axioms D), T), 4), and possibly, adding of RG).

3. The family of B-logics

By **B**-logics we mean all modal logics determined by models with *symmetric* accessibility relation. This group includes not only alethic extensions of **KB**, with super popular **S5**, but also, in a sense, all temporal logics, where symmetry is essential with respect to interrelations between dual past-future operators. This is, no doubt, very important group but, unfortunately, symmetry is extremely hard to deal with in Gentzen systems. Fitting [83] even wrote “Such things (symmetry of accessibility relation) ***effectively destroy*** all possibility of a good, simple, cut-free Gentzen system”.

In what follows we will try to show that things are not as bad as Fitting and other specialists suppose. We sketch how to obtain general cut-free framework for **KB** and its extensions. Simplified account for **S5** was already in Indrzejczak [96], where automatic proof-search procedure and completeness proof based on it are presented. Here we are confined only to the informal explanation of necessary modifications and rationale behind them.

For the rest of this section we think of the proof process as starting with the final sequent and built upward by supplying suitable premises; hence the application of a rule is meant here as leading from conclusion to premise(s). If we get an axiom in every branch, the proof is completed, if not, then we should be able to provide a countermodel on the basis of wffs on this open branch. Of course our **GSC I** as presented in the preceding section is not sufficient for this aim, some modifications are necessary.

At the first sight we could expect that one simple structural rule will do. Add to **GSC-K** a rule that allows to infer $\Gamma \Box \quad \Delta$ from $\Gamma \Diamond \quad \Delta$ and vice versa. With its help we can easily obtain simple proofs of many characteristic **B**-theses. This rule, in fact, allows for a simplification of our apparatus because we can simply cut down on the number of modal sequent types; one is enough. Unfortunately such system is incomplete; for example, we are not able to prove the following **KB**-theses:

$\Diamond\Diamond\Box\Box\phi \rightarrow \phi$ (or more generally $\Diamond^n\Box^n\phi \rightarrow \phi$)
$\Diamond\Box\Box\Box\Box\phi \rightarrow \Box\Box\Diamond\phi$
$\Diamond(\phi 1 \quad \Diamond(\phi 2 \quad \Box(\phi 3 \quad \Box\psi))) \rightarrow \psi$

The source of the problem is quite easy to detect. In all, so far mentioned logics, when in the course of semantic checking the truth of some $\Diamond\phi$ forces us to add some new world to the model, we do not have to bother if it can be the source of some $\Box\psi$, because one is never obliged to look back to worlds which were previously, partially described. What does it mean in syntactic terms of our system? It means here that if we have some C-formula in antecedent (succedent) of modal sequent, we can safely forget about wffs in succedent (antecedent), apply W) to them, then after application of K) and shifting rules, start with $\Diamond$). In other words, structural constraints that $\Diamond$) and $\Box$) apply only to classical sequents, are not too restrictive; there is no danger that we loose some possibility of successful proof search.

In **B**-logics situation is different. It should be possible to apply these rules, even if a sequent is already modal, without loosing one side of a sequent in advance, because in this way we can stop with open branches even if the root-sequent is indeed provable.

In order to capture this effect we have to generalise slightly our concept of modal sequent. But first, some simplifications; instead of formulating and constant application of our structural B-rule, from now on we use in this paragraph only one type of modal sequent, say □ and apply *◇), *□) whenever sequent is modal; TR) does not change the type of a sequent. On the other hand, we introduce the notion of the *grade* of a sequent, hence Γ □□ Δ is of the grade 2, Γ □ Δ of the grade 1, and classical sequent will be treated as a sequent of grade 0. Conventionally, in rules description, we will write simply $\Gamma\ ^{n}\ \Delta$, where $n \geq 0$, to comprise all cases. All the rules are now defined on such sequents, with stipulation that in AX), *) and *) n=0, and in *◇), *□) n=1; rules for □ and ◇ are as follows:

□) $\dfrac{\phi\ ^{n}\ \Delta}{\Box\phi\ ^{n+1}\ \Delta}$ □) $\dfrac{\Gamma\ ^{n+1}\ \phi}{\Gamma\ ^{n}\ \Box\phi}$

◇) $\dfrac{\phi\ ^{n+1}\ \Delta}{\Diamond\phi\ ^{n}\ \Delta}$ ◇) $\dfrac{\Gamma\ ^{n}\ \phi}{\Gamma\ ^{n+1}\ \Diamond\phi}$

Proofs of our examples are now easy to provide; we check the last one:

AX)	$\psi \quad \psi$
□)	$\Box\psi$ □ ψ
W)	$\phi 3$ $\Box\psi$ □ ψ
)	$\phi 3$ $\Box\psi$ □ ψ
□)	$\Box(\phi 3 \ \ \Box\psi)$ □□ ψ
W)	$\phi 2$ $\Box(\phi 3 \ \ \Box\psi)$ □□ ψ
)	$\phi 2$ $\Box(\phi 3 \ \ \Box\psi)$ □□ ψ
◇)	$\Diamond(\phi 2 \ \ \Box(\phi 3 \ \ \Box\psi))$ □ ψ
W)	$\phi 1$ $\Diamond(\phi 2 \ \ \Box(\phi 3 \ \ \Box\psi))$ □ ψ
)	$\phi 1$ $\Diamond(\phi 2 \ \ \Box(\phi 3 \ \ \Box\psi))$ □ ψ
◇)	$\Diamond(\phi 1 \ \ \Diamond(\phi 2 \ \ \Box(\phi 3 \ \ \Box\psi)))$ ψ

*) $\quad -\Diamond(\phi 1 \;\; \Diamond(\phi 2 \;\; \Box(\phi 3 \;\; \Box\psi)))\;\; \psi$

$\to$) $\quad \Diamond(\phi 1 \;\; \Diamond(\phi 2 \;\; \Box(\phi 3 \;\; \Box\psi)))\to\psi$

Soundness still follows by our translation function and we conjecture that it is complete, but to prove completeness we must use other devices than Fitting's Consistency Properties. In Fitting [83] one can find suitable definitions for **KB** and its extensions, but one of the conditions is simply a form of Cut. Adding Cut to our calculus is not a good solution because we do not have, at least for the time being, any proof of its eliminability. The other solution is to define some automatic procedure for proof searching by suitable adjustments in **GSC I**. Generally, any Gentzen system of usual sort needs some reformulations in order to get automatic procedure, hence:

1) we dispense completely with Weakenings, which has the following consequences:

2) we count as axioms all sequents of the form Γ ^{n} Δ, where $\{\phi \;\; -\phi\}$ $\Gamma(\Delta)$, or if n=0, $\Gamma\cap\Delta \neq$.

3) in all two-premises rules the sets of *parameter-formulae* must be unified, hence for example) now reads

$$\frac{\Gamma\ ^{n}\quad \Delta\,\phi \qquad \Gamma\ ^{n}\quad \Delta\,\psi}{\Gamma\ ^{n}\quad \Delta\,\phi\quad \psi}$$

Some other changes will be necessary, specific to our apparatus. We describe them together with the characterization of proof-search procedure. Instead of informative but compact flow-chart we describe them in a more explanatory way. It should go in some stages like these:

0) The input of our search is some sequent Γ Δ, procedure must be so defined that all possibilities of finding out a proof are systematically checked.

1) First of all we must apply all possible classical and shifting rules.

2) If Γ Δ is a genuine modal sequent then after some time we obtain in some branch a sequent

*) $\Gamma'\ \Pi$ $\Delta'\ \Sigma$,

where Π, Σ are C-formulae and Γ', Δ' contain only AT and B-formulae. It is obvious that if this sequent is provable then, immediately above it, some shifting rules and/or W), W) were applied after some $\Diamond$), or $\Box$). But we do not know which wff from Π Σ is the proper candidate, hence we must check them all. Sambin & Valentini [80] and [82] in an analogous situation but for **GL** considered a rule in which we start independently n *subtrees* (not branches) for each of the n C-formulae in a sequent. But in our calculus for **KB** it is not necessary, we simply choose the first of our C-formulae as the candidate for ◇) (□)) and apply shifting rules (no Weakenings!) to get the conclusion of this rule. We do not need to make separate subtrees, at least at this stage, for each C-formula because they are still present in the sequent.

After having done ◇) or □) on our first chosen C-formula, we can have again, a possibility for application of classical and shifting rules (stage 1)), but this time instead of *) and *) we must use *◇) and *□). But once again it is important to modify slightly both rules; now they read:

*◇') $$\frac{\Gamma\ {-}\phi\ \Box\quad \Delta\ \Diamond\phi'}{\Gamma\ \Box\quad \Delta\ \Diamond\phi}$$ *□') $$\frac{\Box\phi'\ \Gamma\ \Box\quad \Delta\ {-}\phi}{\Box\phi\ \Gamma\ \Box\quad \Delta}$$

Here B-formulae are still present in the premise because we should have a possibility for using them again in the new context created after next application of ◇) (□)). On the other hand, we must mark these B-formulae with an apostrophe as *used* in the sequent of this grade, in order not to repeat *◇') and *□') over and again. Note that apostrophes on B-formulae must be deleted whenever we apply ◇) (□)) again so these wffs are marked as used only temporarily.

3) Again, after some time we can face a situation analogous to that of *) from stage 2) but now on modal sequent. In the most general case it may be a sequent of the form

**) $\Gamma\ \Pi\ \Lambda\ ^{n}\quad \Delta\ \Sigma\ \Omega$,

where n > 0, Γ, Δ contain only AT and used B-formulae, Π, Σ are C-formulae and Λ, Ω are non-used B-formulae. This time we cannot simply apply shifting rules in order to transform a sequent into a conclusion of some ◇) (□)) if all displayed sets are non empty. We cannot also apply Weakenings because we do not know in advance which wffs are irrelevant. In procedure we must examine all C-formulae and save all other wffs, because any one may be essential in finding an eventual proof. Hence from **) we must start at most n+k+2 independent subtrees, where n is the cardinality of Π Σ and k of Λ Ω. These are all essential possibilities of applying Weakening together with K), and rules for □ and ◇. We may divide them on 3 groups:

a) If $\Pi \quad \Sigma$ is not empty then we should start 2 subtrees beginning with $\Gamma\ \Pi\ \Lambda$ and $\Delta\ \Sigma\ \Omega$. Both are necessary in order to examine the effect of creating new world by some C-formula from Π (Σ) (stage 2)) and shifting B-formulae to it from an old one (stage 1). Hint: try to prove $\Box(\Box\phi \quad \Diamond\neg\phi)$ without this possibility of continuing proof-search.
a') It is a variant of a); if $\Pi \quad \Sigma$ is empty and n >1 but there are more than 2 B-formulae in Λ (or Ω) then instead of a) we start 2 subtrees beginning with $\Gamma\ \Pi\ \Lambda\ \Box$ and $\Box \quad \Delta\ \Sigma\ \Omega$. They are necessary to check these unused B-formulae in any different world. Hint: try to prove $\Box\Box(\Diamond\phi \quad \Diamond\neg\phi)$ without this possibility.

b) For each B-formula $\Box\phi$ in Λ ($\Diamond\phi$ in Ω) we start a subtree with $\phi^{\,n-1} \quad \Delta\ \Sigma\ \Omega$ ($\Gamma\ \Pi\ \Lambda^{\,n-1} \quad \phi$, respectively) to confront this formula with wffs from the antecedent (succedent). Hint: try to prove $\phi \rightarrow \Box\Box\Diamond\Diamond\phi$.
c) For each C-formula $\Diamond\phi$ in Π ($\Box\phi$ in Σ) we start a subtree with $\phi^{\,n+1} \quad \Delta\ \Sigma\ \Omega$ ($\Gamma\ \Pi\ \Lambda^{\,n+1} \quad \phi$, respectively). Hint: try to prove $\phi \rightarrow \Box(\Box\Diamond\Diamond\phi \quad \psi)$.

In each subtree generated according to a)-c) we proceed further as in stage 1), trying to use classical and shifting rules if possible and repeating stage 3) if necessary. It is also important to delete apostrophes on used B-formulae, at least in subtrees of type b) and c). This general scheme for generating subtrees in stage 3) may be considerably simpler in some cases, e.g., if sets of C- and B-formulae are empty we start only 2 subtrees of a) type. What is important is that there is no need to think of other steps; these are all possible and relevant application of Weakening combined with some other rules that can lead to discovery of a proof. Once again it is important to notice that in stage 3) we are not making new branches but new subtrees (sub- because they are nested in some already created tree). If we apply a branching rule we must end every branch with an axiom in order to obtain a proof, but in stage 3) it is enough to find a proof of at least one premise-sequent of a)-c) in order to have a proof of our concluding-sequent.

Now it should be obvious that it is impossible to find essentially new situation in any subtree, when we proceed further with proof search. So let us examine possible ends of each branch in any subtree. In general our procedure may run ad infinitum (esp. because of b)-subtrees in stage 3)), but because it is deterministic and the number of wffs is finite, we run a loop in this case and may stop. Hence each branch in each subtree eventually must stop either with an axiom or with a sequent $\Gamma^{\,n} \quad \Delta$, where Γ, Δ contain only AT and used B-formulae (if n=0, also unused B-formulae are admissible), in this case stop, no possibility of applying any rule.

If we have at least one elementary subtree (no stage 3) involved) with all branches ended with axioms, then starting sequent is provable. If not, we can construct a countermodel. Briefly, we consider some inductively defined mapping from sequents into N^2 (N for natural numbers). Worlds in the model are signed with all and only those natural numbers which were used as values in the mapping; their number is exactly the number of different C-formulae (not their different occurrences) that appeared in the course of proof-search. Then, starting with non-axiomatic sequent, ending some branch in each subtree we can define a suitable valuation.

This approach easily generalizes to the case of **KBD** and **KBT**. In case of **KB4** (and **S5**) it allows for considerable simplifications; we may use only sequents of the grade 0 and 1 (suitably reformulating rules for $\Box$ and $\Diamond$) and, in consequence, in stage 3) we must consider only subtrees of a) and c) type (a') and b) dispensable). It seems that also temporal logics can be formalized in similar way, but it is the problem for further investigation.

University of Łódź, indrzej@krysia.uni.lodz.pl

REFERENCES

Avron A. [1991] "Simple Consequence Relations", *Information & Computation*, 92, 105-139.

Belnap N. [1982] "Display Logic", *Journal of Philosophical Logic*, 11, 375-417.

Bull R., Segerberg K. [1984] "Basic Modal Logic", [in:] D. Gabbay, F. Guenthner (eds.) *Handbook of Philosophical Logic*, vol. II, Reidel, Dordrecht, 1-88.

Cerrato C. [1990] *Modal sequents for normal modal logics*, typescript, Rome.

Chellas B. F. [1980] *Modal logic*, Cambridge University Press, Cambridge.

Curry H. B. [1952] "The Elimination Theorem when Modality is Present", *Journal of Symbolic Logic*, 17, 249-265.

[1963] *Foundations of Mathematical Logic*, McGraw-Hill, New York.

Dośen K. [1985] "Sequent-systems for modal logic", *Journal of Symbolic Logic*, 50, 149-159.
Feys R. [1950] "Les systèmes formalisés des modalités Aristotéliciennes", *Revue Philosophique de Louvain*, 48, 478-509.
Fitting M. [1973] "Model existence theorems for modal and intuitionistic logics", *Journal of Symbolic Logic*, 38, 613-627.
[1983] *Proof Methods for Modal and Intuitionistic Logics,* Reidel, Dordrecht.
Gentzen G. [1934] "Untersuchungen über das logische Schliessen", *Mathematische Zeitschrift*, 39, 176-210, 405-431.
Indrzejczak A. [1996] "Cut-free sequent calculus for S5", submitted to *Reports on Mathematical Logic*.
Kanger S. [1957] *Provability in Logic,* Almquist & Wiksell, Stockholm.
Mints G. [1970] "Cut-free calculi of the S5 type", *Studies in constructive mathematics and mathematical logic*. Part II. Seminars in Mathematics 8, 79-82.
Ohnishi M., Matsumoto K. [1957] "Gentzen Method in Modal Calculi", *Osaka Mathematical Journal*, 9, 113-130.
Perzanowski J. [1989] "Logiki modalne a filozofia", [in:] J. Perzanowski (ed.) *Jak filozofowac,* PWN, Warszawa.
Sambin G., Valentini S. [1980] "A Modal Sequent Calculus for a Fragment of Arithmetic", *Studia Logica*, 39, 245-256.
[1982] "The modal logic of provability. The sequential approach", *Journal of Philosophical Logic*, 11, 311-342.
Sato M. [1980] "A cut-free Gentzen-type system for the modal logic S5", *Journal of Symbolic Logic*, 45, 67-84.
Smullyan R. M. [1968] *First-Order Logic,* Springer, Berlin.
Trzęsicki K. [1984] "Gentzen-style axiomatization of tense logic", *Bulletin of the Section of Logic*, vol.13/2, 75-84.
Wansing H. [1992] *Sequent Calculi for Normal Modal Propositional Logics*, ILLC Prepublication Series, Amsterdam.
[1995] "Strong Cut-Elimination for Constant Domain First-order S5", *Journal of the IGPL,* 3/5, 797-810.
Zeman J.J. [1973] *Modal Logic,* Clarendon, Oxford.