Logica Trianguli, 1, 1997, 3-14

MODAL LOGIC: A-COMPLETENESS MET UP AGAIN

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Abstract

There are two approaches to logic, semantic and axiomatic. In 1910's, when C.I. Lewis wrote his first papers on modal logic, he adopted the axiomatic approach, the sole one apparently available. The situation remained identical during about forty years, until Kanger, Kripke, and Hintikka (and a few others, about 1955) discovered the so-called *possible worlds* semantics. A new flourishing “paradigmatic” period began, and it became possible to define soundness and completeness in modal logic. Unfortunately (?), however, this period did not go on for more than twenty years. “Strange” systems appeared, which were impossible to be defined semantically and so soundness and completeness were lost again.

1. Fundamental metaproperties

Because the definition of metaproperties such as consistency and completeness have not been totally standardized yet, I shall begin by stating the main ones.

Let S be an axiomatic system, i.e. the ordered set of some formal language, rules of formation of formulas, axioms and rules of deduction. A *theorem* of S is a well formed formula (wff) that can be derived from the axioms by means of the rules of deduction. Let $\mathbb{S}$ be a *semantic framework*, i.e. a set of objects and a procedure allowing to assign the value 1 or 0 to any formula under certain conditions. A *valid* formula of $\mathbb{S}$ is a wff that is assigned 1 under every condition. Let us adopt the well-known abbreviations $\vdash_{S} \alpha$, $\models_{\mathbb{S}} \alpha$ to mean respectively “α is a theorem of S”, “α is valid on $\mathbb{S}$”.

If we consider the propositional calculus (PC) S is e.g. Łukasiewicz' system and $\mathbb{S}$ the machinery of truth-tables. For the first order predicate calculus (PdC) S and $\mathbb{S}$ would be the system and the semantics described by [2] (there are many other equivalent definitions). Finally it may be of interest to speak about arithmetic (Ar), for which S may be the system N studied by Kleene [6] and $\mathbb{S}$ the interpretation considered there.

1.1. Consistency

According to the most current definition S is *consistent* iff there is no α such that $\vdash_S \alpha$ and $\vdash_S \neg\alpha$[1]. *Consistency is a purely axiomatic property*.

PC, PdC are consistent. For Ar Gödel has proved that its consistency is impossible to be proved *by means of its own language*.

1.2. Soundness

S is *sound* (with respect to $\mathbb{S}$) iff every theorem is valid on $\mathbb{S}$, i.e.

$$\vdash_S \alpha \qquad \models_{\mathbb{S}} \alpha$$

Soundness is a property which states a certain relation between an axiomatic system, S, and a semantic framework, $\mathbb{S}$. Proofs of soundness are easy. It suffices to show that every theorem of S is valid on $\mathbb{S}$, and the rules of deduction preserve validity[2].

PC, PdC are sound. The case of Ar is similar to that for consistency.

Soundness implies consistency.

Proof. Suppose S sound but not consistent. Then there is a wff α such that $\vdash_S \alpha$ and $\vdash_S \neg\alpha$. So by soundness we obtain $\models_{\mathbb{S}} \alpha$ and $\models_{\mathbb{S}} \neg\alpha$, which is a contradiction ($\models_{\mathbb{S}} \neg\alpha$ implies $\not\models_{\mathbb{S}} \alpha$, which contradicts $\models_{\mathbb{S}} \alpha$)[3].

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[1] An equivalent formulation reads that no explicit contradiction $\alpha \quad \neg\alpha$ is a theorem. An alternative definition is S is *consistent* iff not every wff is a theorem. Of course, contradictions are derivable in this case and, conversely, in any system admitting Duns Scotus law, $\neg\alpha \to (\alpha \to \beta)$, and the rules of uniform substitution and modus ponens, if α and $\neg\alpha$ are theorems then every wff is a theorem.

[2] Consistency is sometimes named *syntactical* consistency and soundness *semantical* consistency. Thus Kleene, [5] p. 131, writes "The system is *consistent* with respect to the property (or interpretation), if only formulas which have the property (or express true propositions under the interpretation) are provable." If the property at issue is validity, then this means that every provable proposition should be valid, which is soundness.

[3] $\nvdash_S \alpha$ and $\not\models_{\mathbb{S}} \alpha$ obviously denote "not $\vdash_S \alpha$" and "not $\models_{\mathbb{S}} \alpha$".

1.3. Completeness

S is *complete* (with respect to $\mathbb{S}$) iff every wff valid on $\mathbb{S}$ is a theorem of S, i.e.

$$\models_{\mathbb{S}} \alpha \qquad \vdash_{S} \alpha$$

Like soundness, completeness is a property which states a certain relation between an axiomatic system, S, and a semantic framework, $\mathbb{S}$. As known, proofs of completeness are not simple. For the time being, the canonical models method is the best to prove completeness.

PC, PdC are complete, not Ar.

1.4. Adequation or Characterization

This is the conjunction of soundness and completeness. Thus S is *adequate* to, or *characterized* by, $\mathbb{S}$ iff

$$\vdash_{S} \alpha \Leftrightarrow \models_{\mathbb{S}} \alpha$$

In a sense, characterization is the ultimate goal of the logician's activity. Only when characterization is proved, it is possible to use the same letter, e.g. S, to denote theoremhood and validity, that is to use the metasymbol $\models_{S}$.

PC, PdC can be semantically characterized, but Ar cannot.

1.5. Maximality or Saturation

S is *maximal* or *saturated*[4] iff every wff either is a theorem or, joined to S as an axiom, makes it inconsistent (= not consistent). *Maximality is a purely axiomatic property*.

Maximality is sometimes named *strong*, *syntactical* or *absolute* completeness[5], in comparison with the above mentioned completeness, which is then called *weak* or *semantical* completeness.

PC is maximal, but neither PdC nor Ar are maximal.

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[4] The word “maximal” is used in this sense in [4], p. 67. The word “saturated” can be found in Cavaillès' comments upon the axiomatic method (*Œuvres complètes de philosophie des sciences*, Hermann, 1994).

[5] Post, [8], p. 177, describes this property for the propositional calculus (PC), without naming it. Kleene, [6] (French translation, p. 58), calls it "completeness in the sense of Post". Łukasiewicz, [7] p. 82, gives it only the name of "completeness". Church (*Introduction to Mathematical Logic*, 1956, p. 109) refers the property to Post, naming it "completeness". Etc.

The distinction “strong” and “weak”[6] is justified because:

Maximality implies completeness, at least if a semantic framework can be defined with respect to which S is sound.

Proof. Let S+α be S with α added to S as a new axiom. The maximality of S may be expressed as

$$\nvdash_{S} \alpha \qquad S{+}\alpha \text{ inconsistent}$$

or, by contraposition, as

$$S{+}\alpha \text{ consistent} \qquad \vdash_{S} \alpha$$ [7]

To prove the completeness of S, assume that S is sound with respect to a semantic framework, $\mathbb{S}$. Suppose that $\models_{\mathbb{S}} \alpha$. We have to show that $\vdash_{S} \alpha$. Since S is sound

$$\vdash_{S} \beta \qquad \models_{\mathbb{S}} \beta$$

for any β. Now the rules of S preserve validity on $\mathbb{S}$ and we get that S+α is sound, too. Thus,

$$\vdash_{S+\alpha} \gamma \qquad \models_{\mathbb{S}+\alpha} \gamma$$

for every γ. Note that $\mathbb{S}$+α is identical with $\mathbb{S}$, since α is valid on $\mathbb{S}$.

Thus S+α is consistent (soundness implies consistency) and $\vdash_{S} \alpha$.

1.6. Decidability

Here we are not primarily interested in decidability. It is however useful to clarify the matter. The issue being rather complicated, I shall content myself with saying that S is *decidable* iff there exists a *decision procedure* or an *algorithm* for deciding of any wff, in a finite number of steps, whether it is a theorem or not - or whether it is valid or not. In the first case we speak about axiomatic or syntactical decidability, in the second about semantical decidability. When S is characterized, the semantical decidability amounts to the axiomatic one. The truth-table method, e.g., constitutes a decision procedure and proves the semantical decidability of PC and thus also axiomatic.

PC is decidable, but neither PdC nor Ar are.

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[6] They appear in [3], p. 19.

[7] S being consistent, the converse of this meta-implication is true.

Remark 1
Decidability is defined for a *class* of wff, here the wff of S, not only for a single wff. In the case of a given wff, we can define the formal decidability:

A wff α is *formally decidable* iff $\vdash_S \alpha$ or $\vdash_S \neg\alpha$.

This definition offers no advantage for systems like PC or the predicate calculus (PdC), since several wff are trivially not decidable in these systems (e.g. p $\rightarrow$ q, xFx). It is useful only when closed wff do not contain implicit free variable (p, q, F in the example above), for instance in the case of a system formalizing arithmetic. In this case, we define the simply completeness:

S is *simply complete* iff every closed wff is formally decidable.

Thus a system is simply incomplete iff there exists a closed wff that is formally undecidable. If in addition the wff or its negation is valid, then $\models_S \alpha$ $\vdash_S \alpha$ is false and the system is (merely) incomplete. In 1931 Gödel found a true formally undecidable formula of arithmetic and so proved the incompleteness of arithmetic.[8]

Remark 2

One can estimate how much progress has been made in logic in the last seventy years. In 1928 Hilbert believed that "if a formula belonging to arithmetic and not provable in it is added to its axioms, then a contradiction can be derived from the enriched system". This states the maximality of arithmetic. In fact arithmetic is not only not maximal, but also incomplete and consistency is not provable.

2. Metaproperties in modal logic

2.1. Before 1950

If you attempt to build a truth-table for an operator like L = "it is necessary that", you immediately establish that there exists a gap:

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[8] Confusion frequently arises between decidability and formal decidability, or completeness and simply completeness (See e.g. J. Dopp, *Notions de logique formelle*, Nauwelaerts, 1965, p. 78; R. Blanché, *L'axiomatique*, Presses Universitaires de France, 1959, p. 42; G. Chazal, *Éléments de logique formelle*, Hermès, 1996, p. 183).

p	Lp
1	?
0	0

If p is false, then a fortiori it is not necessary. But if p is true, this value does not say by itself whether Lp is true or false.

For other modal operators the situation may be even worse. Consider e.g. the temporal future G = “it will always be the case that”. The truth-table is entirely empty:

p	Gp
1	?
0	?

since whatever the present value of p, that of Gp may be true or false according to circumstances not implied only by the value of p.

Thus at first sight it seems impossible to define a semantics in modal logic, which explains why C.I. Lewis and others began to study modal systems only by the axiomatic approach. An immediate consequence is that only purely axiomatic metaproperties were conceivable, i.e. consistency, maximality, axiomatic decidability.

“Ordinary” systems, T, S4, S5,… appeared to be consistent and not maximal (some are decidable, others not). As usual consistency was easy to prove. Non-maximality was obvious, for no system contains the counterintuitive $p \rightarrow \mathrm{L}p$ as a theorem. Now if this wff is added as an axiom, no contradiction arises since in ordinary systems the converse, $\mathrm{L}p \rightarrow p$, is an axiom and $\mathrm{L}p \leftrightarrow p$ becomes a theorem. In this case the meaning of modality vanishes and, apart from the purely “decorative” L, the system collapses into PC.

Thus before 1950 modal systems were apparently *a-complete*, in that sense that it was impossible to define a semantical completeness. Some authors said “incomplete”, but the word “a-complete” is preferable, since in fact “incomplete” referred to non-maximality, whereas “a-complete” means that completeness *or incompleteness* is by no means definable[9].

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[9] R. Feys, *Modal Logics*, Nauwelaerts, Paris, 1965, wrote under the title "Interpretation" (p. 7): "Modal logic has thus become a highly developed branch of formalized logic. But the question of its interpretation remains open and comes to the foreground".

2.2. The possible worlds "paradigm"

In the middle of 1950's the semantics for modal logic were discovered principally by Kanger, Kripke and Hintikka, the role of Kripke being prominent. They quickly became flourishing. The initial idea of possible worlds stems from Leibniz who said that a necessary proposition is a proposition true in every possible world conceivable by God[10]. A semantical modal framework was conceived as an ordered set of three ingredients. First a set W of "worlds", second a binary relation, R, over the members of W, third a value assignment, V, satisfying certain conditions. Most of these conditions are obvious, at least when one is aware of the fact that valuations of purely propositional functions are made separately in each world. For instance:

> For any wff, α, and any w W, $V(\neg\alpha, w) = 1$ if $V(\alpha, w) = 0$; otherwise $V(\neg\alpha, w) = 0$.

But of course the vital condition applies to the modal operator L:

> For any wff, α, and any w W, $V(L\alpha, w) = 1$ if for every w' W such that wRw', $V(\alpha, w') = 1$; otherwise $V(L\alpha, w) = 0$.

As in any other semantical framework, a wff, α, is valid iff it is true in every world of every model (i.e. for every model $\langle W,R,V \rangle$ and for every w W, $V(\alpha, w) = 1$).

Comparing this semantics with Leibniz' primitive idea, perhaps one could say that two important modifications have been made. The former is the cancellation of God, the latter (and more important) is the introduction of the *accessibility relation*, R, thanks to which a great plurality of systems can be defined, depending on the semantical properties intended to apply to it. For instance, semantics for the systems K or T, S4, S5 are obtained if R respectively has no property or is reflexive,

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The most obvious interpretation remains that intended by Lewis, identifying necessity with "logical necessity", which can be characterized syntactically. […]". At this date Feys was able to be aware of the semantics of possible worlds. But he apparently did not realize its importance.

[10] See e.g. *Théodicée*, I, 8-9 ("J'appelle monde toute la suite et toute la collection de toutes les choses existantes").

or reflexive and transitive, or reflexive, symmetric and transitive. As it is well known, these systems are axiomatically defined by[11]:

K	=	PC + **K** + **N**
T	=	K + **T**
S4	=	**T** + **4**
S5	=	**T** + **5**

where

N	is	$\vdash \alpha \quad\quad \vdash \mathrm{L}\alpha$
K		$\mathrm{L}(p \rightarrow q) \rightarrow (\mathrm{L}p \rightarrow \mathrm{L}q)$
T		$\mathrm{L}p \rightarrow p$
4		$\mathrm{L}p \rightarrow \mathrm{LL}p$
5		$\neg\mathrm{L}p \rightarrow \mathrm{L}\neg\mathrm{L}p$

Let us now consider the two most important properties, soundness and completeness[12]. As usual the former is easy to prove (see above) and the latter is elegantly solved by the canonical models method. In this method a "world" consists of maximal consistent sets of wff. In a system S a set, Λ, of wff is *consistent*[13] iff there is no $\alpha_1, \ldots, \alpha_n \quad \Lambda$ such that

$$\vdash_{\mathrm{S}} \neg(\alpha_1 \quad \ldots \quad \alpha_n)$$

in other words no subset of Λ can be proved to be contradictory.
A set , Γ, of wff is maximal iff for every wff α either $\alpha \quad \Gamma$ or $\neg\alpha \quad \Gamma$.
Maximal consistent[14] sets present "good" features like:

$\alpha \quad \beta \quad \Gamma$ iff $\alpha \quad \Gamma$ and $\beta \quad \Gamma$

and Lindenbaum's theorem[15] says that, for a given system, S

If Λ is a consistent set of wff, then there is a maximal consistent

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[11] As [4] I shall use bold-face type when referring to an axiom, but roman type when referring to a system. So the *system* K cannot be confused with the axiom **K**.

[12] I set aside the other properties for the following reasons. Consistency is an immediate consequence of soundness. Maximality never holds for modal systems (except for Triv and Ver). Decidability gives rise to complicated problems I prefer not to get onto here.

[13] Of course the consistency of a set of wff should not be confused with that of a system.

[14] Of course again this maximality for a set of wff should not be confused with that of a system.

[15] See [1], p. 55.

set, Γ, of wff such that $\Lambda \quad \Gamma$.

Thus worlds of the canonical model for a system S can always be constructed and, by definition, any propositional variable is true in a canonical world iff it is a member of that world. As a consequence of the properties mentioned above, a formula is true in a canonical world iff it is a member of that world. Thus a wff is valid in the canonical model iff it is true in its every world, i.e. iff it is a member of every world in the model (CS denoting the canonical model for S):

$$\models_{CS} \alpha \Leftrightarrow \text{ for every } \Gamma \ \ \alpha \quad \Gamma$$

On the other hand it is readily proved that canonicity and maximality entail

$$\vdash_{S} \alpha \Leftrightarrow \text{ for every } \Gamma \ \ \alpha \quad \Gamma$$

From which

$$\models_{CS} \alpha \Leftrightarrow \vdash_{S} \alpha$$

Finally if we can prove that the canonical model is included in a certain class $\mathbb{S}$ of models (e.g., for T, those with a reflexive accessibility relation), then every wff valid in the models of this class will be a theorem. This is completeness

$$\models_{\mathbb{S}} \alpha \qquad \vdash_{S} \alpha$$

When soundness and completeness became definable for modal logics, they recovered “normality” (every system containing K is named *normal*). Paraphrasing T.S. Kuhn, we could say that modal logic entered a paradigmatic period. That period lasted about twenty years until the concept of frame was discovered.

2.3. The rationalization created by frames

Delete the value assignment, V, in a model <W, R, V>, i.e. consider the ordered set <W, R>. You obtain a *frame*. It seems a good idea to introduce this concept since modal logic deals with pluralities of worlds and the truth-values of wff in the different worlds may change. In that sense a frame is more fundamental than a model. Before frames were explicitly introduced, they were implicitly present. When validity is defined as truth in a class of models, that class is in fact a frame. But using only models nothing would prevent us from defining special classes of models, for instance a class where every propositional variable is true or a class where every propositional variable is false, etc. Validity in this case would be very artificial since we conceive the “right” concept as truth under every circumstance. In other words, as is the case of PC, validity must represent preservation of truth when the initial values of the value assignment, V, change as much as possible. Deleting V, frames just represent the rational choice for classes of models. “Exotic” models are excluded.

2.4. The first “anomaly”: the system KW

Let us define *a frame for* S as a frame on which every theorem of S is valid (i.e. true in every world in every model based on it)[16]. S is said to be *canonical* iff the frame of S's canonical model is a frame for S[17].
For instance, it can be proved that:

(i) the accessibility relation of the canonical model of T is reflexive;

(ii) every theorem of T is valid in any model based on reflexive frames.

Thus by (ii) every reflexive frame is a frame for T and therefore by (i) T is canonical.
Let KW be the system K+**W**, where

W $\quad L(Lp \rightarrow p) \rightarrow Lp$

KW is not canonical. This can be proved by showing that[18]:

(i) the frame of the canonical model of KW contains at least one world accessible to itself;

(ii) **W**, although valid in the canonical model as is the case for every theorem of KW, is not valid on every frame which contains a world accessible to itself.

Thus a great difficulty apparently arises. Obviously, the canonical model must belong to every frame for KW, but the frame containing the canonical model is not a frame for KW.
Clearly there is only one solution to this dilemma, that is changing the canonical model itself.

A way of doing that consists in restricting canonical models to finite canonical ones. Defining Ψ_α as the set of all sub-formulas of α and their negations, we can construct a finite canonical model, i.e. with a finite number of worlds. Obviously its corresponding frame is finite. Then it can be proved that KW is complete with respect to the class of finite, irreflexive and transitive frames[19]. As stressed by Hughes & Cresswell [4], finiteness is here crucial:

“The system characterized by *all* transitive irreflexive

[16] [4], p. 172.
[17] *Ibid.*, p. 140.
[18] *Ibid.*, p. 139-141.
[19] *Ibid.*, p. 150.

frames is K4. That does not mean that K4 lacks the finite model property[20] […]. But although K4 is characterized by the class of all finite transitive frames and by the class of all transitive and irreflexive frames it is not characterized by any class of finite transitive and irreflexive frames.”

2.5. The second “anomaly”: the system KH

Consider now the system K+**H**, where

H $\quad L(Lp \leftrightarrow p) \rightarrow Lp$

The situation is dramatically worse than for KW. KH is characterized by no class of frames at all. This can be proved by showing that:

(*) if **H** is valid on a frame F, so is $Lp \rightarrow LLp$;

(**) $Lp \rightarrow LLp$ is not a theorem of KH.

Without specifying any detail, it is noteworthy that the proof is possible because (*) requires the use of a *frame*, whereas (**) is based “only” on a *model* (where $Lp \rightarrow LLp$ is false and every instance of H is valid[21]). Thus the impossibility to define a semantics characterizing KH rests on the one hand upon the difference between frames and models. On the other hand the problem of finiteness plays an important role. It can be shown that a system characterized by a class of finite models is also characterized by a class of finite frames. Thus infiniteness is indispensable for the “strangeness” of KH. Hughes & Cresswell [4] suggest that we should refer to the impossibility of characterizing semantically KH by using the term *incompleteness*. But a system S must be called incomplete with respect to some class of frames, **C**, iff there is at least a formula valid on the frames of **C** which is not a theorem of S. Of course with respect to “stronger” semantics (e.g. that for KW) KH is incomplete, but in fact the problem is that there is no semantics at all which can characterize KH. Perhaps then the word *a-completeness* would be more appropriate.

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[20] A system is said to have the finite model property when soundness and completeness can be proved with respect to a class of finite frames.

[21] See [4], p. 160ff. According to theorem 9.1, when every instance of a formula α is valid, then every theorem of K+α is valid). Note that the system characterized by the class of all frames for KH is KW.

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