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Duality, paraconsistency and inferential logics

Abstract. Inferential counterparts of Tarski's structural logics are set on a non-reflexive and non-idempotent q-consequence operations. Their q-matrix semantics is based on two non-complementary categories of rejection and acceptance, what makes the dualization natural. In the paper we also discuss the paraconsistency properties of inferential logics, what in view of duality seems to be interesting from the point of possible application of the inferential framework as a methodology for information processing.

Keywords: duality, negation, paraconsistency, consequence, q-consequence, many-valued logic, structurality.

1. Inferential logics

A *structural logic* for $\underline{L}$ is a pair $(\underline{L}, C)$, where $\underline{L} = (For, f_1, f_2, ..., f_n)$ is a sentential language and $C : 2^{For} \rightarrow 2^{For}$ the Tarski's reflexive, monotonic and idempotent consequence operation: for any sets of formulas X, Y

$$X \subseteq C(X) \qquad (refl)$$
$$C(X) \subseteq C(Y) \text{ whenever } X \subseteq Y \qquad (mon)$$
$$C(C(X)) = C(X), \ \textit{and} \qquad (cl)$$
$$eC(X) \subseteq C(eX), \text{ for every substitution } e \text{ of } \underline{L}. \qquad (st)$$

The revision of these axioms motivated by the mathematical practice treats *auxiliary assumptions* as mere hypotheses, accounted for by deduction (or not), which justifies their occurrence in the place of conclusions. To this aim we neglected (*refl*) and weakened the closure condition (*cl*), cf. [4].

The central semantical notions of the inferential framework are: a "quasi" counterparts of matrix, a *q-matrix*, and matrix *q-consequence*. Where $\underline{L}$ is a sentential language and $\underline{A}$ an algebra similar to $\underline{L}$, a *q-matrix* is a triple

$$M^* = (\underline{A}, D^*, D),$$

where D^* and D are disjoint subsets of A of *rejected* and *accepted* elements, respectively. The relation $\vdash_{M^*} \subseteq 2^{For} \rightarrow For$, of a *matrix q-consequence of* M^*, defined by

$$X \vdash_{M^*} \alpha \text{ iff for every } h \in Hom(\underline{L}, \underline{A})(h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset).$$

is a formal counterpart of reasoning admitting rules of inference which from non-rejected assumptions lead to accepted conclusions.

The q-concepts and usual concepts of matrix and consequence coincide when $D^* \cup D = A$, i.e. when D^* and D are complementary. Then, the set of rejected elements is the set of non-accepted elements. If, however, the sets D^* and D are not complementary we obtain the partition of the set A of original matrix values into three disjoint subsets: D^*, $A \smallsetminus (D^* \cup D)$, D.

An arbitrary structural q-consequence $Q : 2^{For} \to 2^{For}$, defined through the following conditions:

$$Q(X) \subseteq Q(Y) \text{ whenever } X \subseteq Y \tag{mon}$$

$$Q(X \cup Q(X)) = Q(X) \qquad \text{and} \tag{qcl}$$

$$eQ(X) \subseteq Q(eX), \text{ for every substitution } e \text{ of } \underline{L}. \tag{st}$$

Obviously, the operation $Qn_{M^*} : 2^{For} \to 2^{For}$:

$$Qn^*_M(X) = \{\alpha : X \vdash^*_M \alpha\},$$

associated with a given q-consequence relation $\vdash^*_M$ satisfies (mon), (qlc) and (st) and is called a *matrix q-consequence operation* of M^*.

Whenever a given structural q-consequence Q is not a consequence, the logic (L, Q) is called $inferential$ since all rules of Q are given explicitly, i.e. none of them is given through a methodological postulate. Notice that it is no case a consequence C since the unlimited rule of repetition follows from the Tarski's reflexivity postulate (*refl*).

2. Dualization nad paraconsistency

$Dualization.$ Getting the dual counterpart of a given inferential logic is easy and it reduces to a simple operation on q-matrices. Due to the completeness theorem for inferential logics, cf. [4], stating that for every structural inference operation Q there is a class K of inferential matrices such that

$$Qn_K(X) = \bigcap\{Wn_{M^*}(X) : M^* \in K\}.$$

Further to this, to get natural dual to a q-matrix

$$M^* = (\underline{A}, D^*, D),$$

it suffices mutually change the roles of two designated sets of elements. Accordingly,the dual q-matrix

$$dM^* = (\underline{A}, D, D^*),$$

generates an inference from non-accepted premisses to rejected conclusions.

Clearly, the dual dQ to a given structural q-consequence $Q = Qn_K$ coincides with dQn_K, where

$$dQn_K(X) = \bigcap\{Wn_{dM^*}(X) : M^* \in K\}.$$

Paraconsistency. Paraconsistent logics are constructed or discovered as systems of theorems and rules defining a non-explosive consequence relation: $\vdash$ is explosive whenever inconsistency implies overcompleteness, i.e. when $\{\alpha, \neg\alpha\} \vdash \beta$, for any α, $\beta \in For$.

Inferential logics and inferential conservative extensions of some "standard" logics happen to have paraconsistency-like properties independent from connectives. To start with, consider sentential language $\underline{L}$, which among its connectives has a negation, $\neg$. Let, subsequently, take any q-matrix $M^* = (\underline{A}, D^*, D)$ for $\underline{L}$, and assume that the set $I = A \smallsetminus (D^* \cup D)$ is closed under the negation, i.e. $\neg a \in I$, whenever $a \in I$. Notice then, that the matrix inference $\vdash_{M^*}$ is (inferentially) paraconsistent: it is not true, that $\{p, \neg p\} \vdash q$. To verify this it suffices to take any valuation, which sends p to I, and q to D.

The main source of *inferential paraconsistency* so introduced is just the inference, and not the language, as in case of the widely exploited concept. Alas, some logics not paraconsistent *natively*, with respect to their primitive connectives, may have the inferential property.

3. Kleene and Łukasiewicz inferential logics

Kleene. The three-valued Kleene logic K_3, based on the values f, u, t, representing: falsity (f), undefiniteness (u), and (truth) t. Since the content of the Kleene matrix is empty, i.e. it does not have "tautologies", the system is not paraconsistent natively. Now, if take the referential extension of K_3 based on the q-matrix of the form $(\{f, u, t\}, \sim, \to, \vee, \wedge, \{f\}, \{t\})$ defines inferentially paraconsistent inference: it is not true, that $\{p, \neg p\} \vdash q$. The latter property is due to the fact that the set $\{u\}$ is closed under negation: according to the truth table of negation: $\neg u = u$, cf. [3]

Łukasiewicz. $\underline{L}_n = (L_n, \neg, \rightarrow, \vee, \wedge, \equiv)$ will denote the algebra of Łukasiewicz matrix: for a finite $n \geq 2$, $L_n = \{0, {}^1\!/_{n-1}, {}^2\!/_{n-1}, \ldots, 1\}$,
for $n = \aleph_0, L_n = \{s/w : 0 \leq s \leq w,\ s,\ w \in N \text{ and } w \neq 0\}$
for $n = \aleph_1, L_n = [0, 1]$,
and the functions for the connectives: unary $\neg$ (for *negation*) and binary $\rightarrow$, $\vee$, $\wedge$, $\equiv$ (for *implication, disjunction, conjunction* and *equivalence*) are defined on L_n by the equalities: $\neg x = 1 - x$, $x \rightarrow y = min(1, 1 - x + y)$, $x \vee y = max(x, y)$, $x \wedge y = min(x, y)$, $x \equiv y = 1 - |x - y|$.

All *Łukasiewicz matrices* have the same form $M_n = (\underline{L}_n, \{1\})$, The natural extension of M_n to a q-matrix is received when keeping 1 as the only accepted element and taking 0 as the single rejected value. We then get the q-matrix

$$M_n^* = (\underline{L}_n, \{0\}, \{1\}) ,$$

that may be used to define the consequence relation verifying inferences, which from non-false premisses yield the truth conclusions. Note, that the content of this q-matrix coincides with the content of M_n, i.e. is the set of all tautologies of Łukasiewicz system.

Since any n-valued Łukasiewicz logic is complete with respect to M_n, we may check its properties directly, using the matrix. So, is it easy to verify that the formula $p \rightarrow (\neg p \rightarrow q)$ is a tautology of the system, and that the detachment rule is a rule of the logic. By this, we get that $\{p, \neg p\} \vdash q$. Accordingly, all Łukasiewicz logics are explosive and thus not paraconsistent *natively*, i.e. with respect to their primitive connectives.[1] On the other hand, we have another argument at hand: the set $I_n = L_n \smallsetminus (\{0\} \cup \{1\})$ is closed under the negation, i.e. $\neg a \in I_n$, whenever $a \in I_n$. Therefore, any valuation, which sends p to I_n, and q to $\{0\}$, falsifies $\{p, \neg p\} \vdash q$. Thus, all Łukasiewicz logics are inferentially paraconsistent.

4. Postscript

Forming dual counterparts of logics considered in Section 2 is easy. The semantics of all these logics reduces to a single matrices and mutual change of the rôles of designated sets is sufficient for getting corresponding dual q-matrices, adequate to their dual counterparts. So e.g., a given Łukasiewicz's M_n^* dual

[1] da Costa and Kotas showed in [1] that all finite Łukasiewicz logic are paraconsistent with respect to some definable connectives.

$$dM_n^* = (\underline{L}_n,\ \{1\},\ \{0\})\ .$$

generates the inference from non-true premisses to false conclusions. It is not surprising, though, that the content of this matrix coincides with the set of all negations of Łukasiewicz system.

In the end, remark that for the dual Kleene and Łukasiewicz inferential logics one may formulate *paraconsistency* problem and give a satisfying answer to it. Also, in these cases some sets $\{\alpha, \neg\alpha\}$ are not explosive. So, as before, it is not true that $\{p, \neg p\} \vdash q$, even if the roles of truth and falsity dually interchanged. More thorough inspection of the dual properties sheds some light on the nature of paraconsistency as well as on the way of naming its two modes, see [2].

References

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