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CONCERNING INTUITIONS ON LOGICAL MANY-VALUEDNESS

Abstract

The widely acclaimed Suszko's Thesis on logical two-valuedness of structural logics raised a question on whether it is possible to define a relation of inference that would not have description in terms of zero-one valuations. A positive answer to this question was possible in the result of a division of the logic algebra of sentence referents into three subsets of elements: rejected, accepted, and neutral. The division affords possibilities to define a natural inference using the rules of inference which from non-rejected premises lead to the accepted conclusions. Accordingly, Tarski consequence is weakened to a q-consequence, see [6], the most cases of which are logically three-valued, since they can only be described by means of three-element valuations.

Our aim is to discuss logically three-valued q-consequence matrix-based operations that have representation in terms of two-valued consequence operations. The possibility of such representation sheds new light on logical three-valuedness and calls for further exploration of non-orthodox inference operations possible – it was already used, e.g. for the construction of n-valued matrix inferences, for $n \geq 4$, see [4]. We also believe, that such an approach is a good start to the investigation of potentially rich spectrum of possible non-classical inferences, founded on linear and nonlinear q-matrices.

The present paper is a version of the lecture given to the 2^{nd} *Conference on Non-classical Logic. Theory and aplication*, Łódź, 17-19 September 2009. The talk was a tribute to Roman Suszko, whose ideas influenced my way of thinking on inferential many-valuedness.

Keywords: consequence operation, q-consequence, many-valuedness, three-valuedness, logical value, inferential value, inferential logic, logical two-valuedness, Suszko's Thesis, logical n-valuedness, Kleene logic.

1. Structurality and logical three-valuedness

A structural logic in a sentential language $\underline{L}$ is a reflexive, monotonic and idempotent Tarski's *consequence operation* $C : 2^{For} \to 2^{For}$,

(*ref*) $X \subseteq C(X)$

(*mon*) $C(X) \subseteq C(Y)$ whenever $X \subseteq Y$

(*cl*) $C(C(X)) = C(X)$, and

(*st*) $eC(X) \subseteq C(eX)$, for any substitution e of $\underline{L}$.

The main motivation for the Tarski's axioms revision comes from the mathematical practice of treating some auxiliary assumptions as mere hypotheses. These assumptions, depending on the rules of a deduction, may be (or not) accounted for as assumptions of further inference. This led us to question the reflexivity postulate and, consequently, to weaken the closure condition (T2), see [6].

On semantic grounds this results in the tripartition of the matrix universe into sets of rejected, neutral and accepted elements. The appropriate modelling relies on matrices with two distinguished sets of elements. A *q-matrix* for a given sentential language is a triple

$$M^* = (\underline{A}, D^*,\ D),$$

where $\underline{A}$ is an algebra similar to $\underline{L}$, while $D^*$and D are disjoint subsets of A, the universe of $\underline{A}$, interpreted as rejected and accepted elements, respectively. A relation $\vdash_{M^*}$ is said to be a *matrix q-consequence of* M^* provided that for any $X \subseteq For, \alpha \in For$

$$X \vdash_{M^*} \alpha \text{ iff for every } h \in Hom(\underline{L}, \underline{A})(h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset).$$

$\vdash_{M^*}$ was set as a formal counterpart of reasoning admitting the rules of inference which from non-rejected assumptions lead to accepted conclusions.

The q-concepts coincide with standard concepts of matrix and consequence only if $D^* \cup D = A$, i.e. when D^* and D are complementary. Then, the set of rejected elements coincides with the set of non-accepted elements.

Putting $D^0 = A - D^*$ conforms q-matrix and the matrix q-consequence to our intuitions even better:

$$M^0 = (\underline{A}, D^0, D),$$

$$E(M^0) = \{\alpha \in For : h\alpha \in D \text{ for any } h \in Hom(\underline{L}, \underline{A})\},$$

$$X \vdash_{M^0} \alpha \text{ iff for every } h \in Hom(\underline{L}, \underline{A}) \text{ (If } h(X) \subseteq D^0\text{, then } h(\alpha) \in D),$$

since then there are two distinguished sets of elements: one for premises - D^0, and one for conclusions - D. Now, $D \subseteq D^0$ and while the former is still the set of accepted elements, the latter is the set of non-rejected elements. Obviously, if D and D^0 are equal, the q-concepts and the usual concepts of matrix and consequence coincide. However, always $E(M^0) = E(M)$, where $M = (\underline{A}, D)$ and that means that both logics of inferences (with Tarski consequence and with the q-consequence) extend the same set of tautologies.

An operation $Qn_{M^0} : 2^{For} \to 2^{For}$ such that

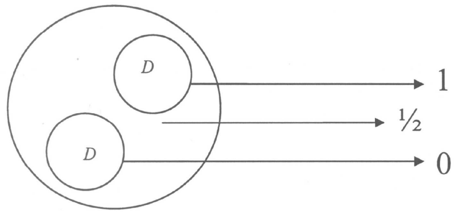
$$Qn_{M^0}(X) = \{\alpha : X \vdash_{M^0} \alpha\}$$

associated with $\vdash_{M^0}$ is called a *matrix q-consequence operation* of M^0.

The binary description of the matrix q-consequence relation is not, in general, possible. Instead, one may define a three-valued valuation function $k_h : For \to \{0, {}^1\!/_2, 1\}$ putting

$$k_h(\alpha) = \begin{cases} 0 & \text{if } \; h(\alpha) \in D^* \\ {}^1\!/_2 & \text{if } \; h(\alpha) \in A - (D^* \cup D) \\ 1 & \text{if } \; h(\alpha) \in D, \end{cases}$$

for $h \in Hom(\underline{L}, \underline{A})$. Notice that the valuations k_h are "three-valued" characteristic functions of two subsets of matrix values: rejected and accepted:



Now, consider the class of all valuations $KV^*_M = \{k_h : h \in Hom(\underline{L}, M^*)\}$, it allows for three-valued description of the matrix q-consequence of $\vdash_{M^*}$:

$$X \vdash_{M^*} \alpha \text{ iff for every } k_h \in KV_{M^*} (k_h(X) \cap \{0\} = \emptyset \text{ implies } k_h(\alpha) = 1).$$

It easily follows from the properties of q-consequence that the description is not, in general, reducible to the two-valued. Thus, the latter property can be interpreted as the logical two-valuedness, we may say that a q-logic $\vdash_{M^*}$ is logically either two or three valued. The property extends, via Lindenbaums representation onto an arbitrary structural q-consequence $Q : 2^{For} \to 2^{For}$, defined through the following conditions:

$$Q(X) \subseteq Q(Y) \text{ whenever } X \subseteq Y \qquad (mon)$$

$$Q(X \cup Q(X)) = Q(X) \qquad \text{and} \qquad (qcl)$$

$$eQ(X) \subseteq Q(eX), \text{ for every substitution } e \text{ of } \underline{L}. \qquad (st)$$

Remark. Any structure

$$Ł^3_q = (\{0, {}^1\!/_2, 1\}, \sim, \Rightarrow, \vee, \wedge, \equiv, \{0\}, \{1\}),$$

may serve an an example of a matrix whose q-consequence is logically three-valued: it is not true that $\{p\} \vdash_{M^*} p$, any $p \in Var$. To see this, take a valuation sending p into ${}^1\!/_2$.

2. Finite linear matrices and q-matrices

In [4] some finite linear logic algebras and q-matrices with the connectives labelling "appropriate" intervals of their universe were constructed. Among the reasons behind their construction there were the two following: (1) settling intuitive grounds for inferential n-valuednes, and (2) building a family of logics logically n-valued for $n \geq 4$. Below, we present that auxilary construction in details.

For a finite natural $n \geq 2$ let $E_n = \{1, 2, 3, \ldots, n\}$ be the universe of a logic algebra $\underline{E}_n = (E_n, f_1, f_2, \ldots, f_m)$ for a sentential language $\underline{L} = (For, F_1, \ldots, F_m)$. Let $D^i = \{i, i+1, \ldots, n\}$, $i \in \{1, 2, \ldots, n\}$. Assume that among the functions of connectives of $\underline{E}_n$, primitive or definable, there are unary functions $\delta_1, \delta_2, \ldots, \delta_n$:[1]

$$\delta_i(x) = \begin{cases} n & \text{if} \quad x \geq i \\ 1 & \text{if} \quad x < i. \end{cases}$$

Let $D^k = \{k, k+1, \ldots, n\}$ and $D^l = \{l, l+1, \ldots, n\}$, for $k < l$ and let

$$M^{k,l} = (\underline{E}_n, D^k, D^l),$$

Consider its q-inference,

$$X \vdash_{M^{k,l}} \alpha \text{ iff for every } h \in Hom(\underline{L}, \underline{E}_n) \text{ (If } h(X) \subseteq D^k, \text{ then } h(\alpha) \in D^l),$$

and its content,

$$E(M^{k,l}) = \{\alpha : h(\alpha) \in D^l \text{ for every } h \in Hom(\underline{L}, \underline{E}_n)\}.$$

Next, take also the matrix $M^n = (\underline{E}_n, \{n\})$, its consequence $\models^n$

$$X \models^n \alpha \text{ iff for every } h \in Hom(\underline{L}, \underline{E}_n) \text{ (If } h(X) \subseteq \{n\}, \text{ then } h(\alpha) \in \{n\})$$

and its content

$$E(M^n) = \{\alpha : h(\alpha) = n \text{ for every } h \in Hom(\underline{L}, \underline{E}_n)\}$$

Let $\delta_k(X) =_{df} \{\delta_k(\beta) : \beta \in X\}$. Notice that

(•) $$X \vdash_{M^{k,l}} \alpha \text{ iff } \delta_k(X) \models^n \delta_l(\alpha),$$

and, finally, that

$$Qn_{M^{k,l}}(X) = \{\alpha : X \vdash_{M^{k,l}} \alpha\} = \{\alpha : \delta_k(X) \models^n \delta_l(\alpha)\}.$$

The last equation shows two descriptions of sets $Qn_{M^{k,l}}(X)$ and thus, descriptions of the matrix q-consequence operation $Qn_{M^{k,l}}$.

[1] Such functions are definable in standard logics, cf. [7], a family of which contains most of the known finite-valued calculi, see Section 3. They are also widely used in modern formulations of Post algebras.

Corollary. The second description may be seen as a representation of the q-consequence $Qn_{M^{k,l}}$ in terms of a "partial" consequence operation Qn_{M^n}. Accordingly,

$$\alpha \in Qn_{M^{k,l}}(X) \equiv \delta_l(\alpha) \in Qn_{M^n}(\delta_k(X)).$$

3. Standard q-consequence case

Rosser and Turquette [7] in their thorough analysis of (referential) many-valuedness settled some requirements for logic algebras, matrices and the logical connectives, which make their many-valued constructions similar to the classical logic. Rosser and Turquette considered matrices of the form

$$M^k = (\underline{E}_n, D^k),$$

having the so-called *standard connectives* of negation ($\neg$), implication ($\rightarrow$), disjunction ($\vee$), conjunction ($\wedge$), equivalence ($\equiv$) and the functions j_1, j_2, ... , j_n "dentifying" logical values, respectively:

$$j_i(x) = \begin{cases} n & \text{if} \quad x = i \\ 1 & \text{if} \quad x \neq i. \end{cases}$$

All other standard connectives behave, with respect to the set of designated elements D^k and its complement $E_n - D^k$, in a similar way as their classical counterparts with respect to truth and falsity, here represented by n and 1, respectively. Thus, we have the *standard conditions* for the functions of negation, implication, disjunction and conjunction:

$$\begin{array}{lll} \neg x \in D^k & \text{iff} & x \notin E_n - D^k \\ x \rightarrow y \notin D^k & \text{iff} & x \in D^k \text{ and } y \notin D^k \\ x \vee y \notin D^k & \text{iff} & x \notin D^k \text{ and } y \notin D^k \\ x \wedge y \in D^k & \text{iff} & x \in D^k \text{ and } y \in D^k. \end{array}$$

Taking this into account, we immediately get that the functions $\delta_1, \delta_2, \ldots, \delta_n$ from Section 2 are definable in standard matrices. Thus, for any $x \in E_n$

$$\delta_i(x) = j_i(x) \vee j_{i+1}(x) \vee \ldots \vee j_n(x)$$

Therefore, the standard matrices of Rosser and Turquette are special linear matrices considered in Section 2 and all *standard q-matrices* behave alike. In what follows we shall show that, in the standard environment, the essential properties of q-inferences may be further reduced and they may be even expressed in terms of tautologies, i.e. the formulas in the content of matrices.

To this aim, for any finite set of s formulas, $X_f = \{\beta_1, \beta_2, ..., \beta_s\}$, consider the conjunction $\bigwedge(\delta_i(X_f)$ of $\delta_i(X_f) = \{\delta_i(\beta_1), \delta_i(\beta_2), ..., \delta_i(\beta_s)\}$, i.e. the formula

$$\delta_i(\beta_1) \wedge \delta_i(\beta_2) \wedge ... \wedge \delta_i(\beta_s).$$

Now, let us consider standard matrix M^n and any "standard" q-matrix $M^{k,l}$ related to M^n. Since M^n is finite, the relation $\models^n$ is finitary and therefore, the condition

(i) $\delta_k(X) \models^n \delta_l(\alpha)$

is equivalent to

(ii) for some finite $X_f \subseteq X$, $\delta_k(X_f) \models^n \delta_l(\alpha)$.

Subsequently, due to the fact that $\models^n$ has the deduction theorem with respect to $\rightarrow$ and $\wedge$, see [7], [8], we may replace (ii) with

(iii) for some $X_f \subseteq X$, $\emptyset \models \bigwedge \delta_k(X_f) \rightarrow \delta_l(\alpha)$

or, by

(iv) for some finite $X_f \subseteq X$, $(\bigwedge \delta_k(X_f) \rightarrow \delta_l(\alpha)) \in E(M^n)$

The equivalence (i) $\equiv$ (iv) and ($\bullet$) in Section 2 yield that for the standard $M^{k,l}$, $X \vdash_{M^{k,l}} \alpha$ is equivalent to (iv). From this we, finally, get the ultimate description of the q-consequence operation $Qn_{M^{k,l}}$ using the content of the "generic" standard matrix M^n or, more precisely, by a set of its selected tautologies:

$$Qn_{M^{k,l}}(X) = \{\alpha : \text{ for some finite } X_f \subseteq X, (\bigwedge \delta_k(X_f) \rightarrow \delta_l(\alpha)) \in E(M^n).$$

The family of standard logics of Rosser and Turquette is quite large and it contains several known finite -valued logics, including the time honoured Łukasiewicz and Post calculi. On the other hand, the special δ functions (or connectives) play a very important role in modern algebraic characterisation, description and formulation of finite many-valued logics. Some of their use already led to the excelent applications.

4. Inferential logic for empirical discourse

The technique of representation of q-consequence in terms of consequence may be also applied to some other logics founded on matrices. One example of the kind is an inferential extension of the Kleene strong logic based on three values: falsity (f), undefiniteness (u) and truth (t), see [2]. The basis of that extension was Körner's attempt of applying the classical logic to the empirical discourse. The core of the approach in [3] was to accept the inexactness and neutrality of the sentences of the "third category". Körner extended the Kleene logic with two more connectives, of strong and weak idealisation. Even if his research programme failed[2], his construction is very important in several aspects. In Section 2 of [5] the concept of an empirical inference and its properties were extensively discussed. It was shown that, to some extent, the Körner's proposal may be conceived as a logic for empirical discourse. Below, we return to the problem for the purpose of giving another example of a "clasically" reducible q-consequence.

The language $\underline{L}$ on which the empirical inference is defined contains the strong Kleene counterparts of standard connectives of negation ($\sim$), implication ($\Rightarrow$), disjunction ($\vee$), conjunction ($\wedge$) and equivalence ($\Leftrightarrow$). Additionally, it has the connectives expressing strong and weak idealisation of undefinite empirical sentences: $S(trong)$, turning undefinite sentences into true, and $W(eak)$, turning undefinite sentences into false sentences. Therefore, the "empirical" logic algebra has the form

$$\underline{A}_3 = (\{f, u, t\}, \sim, \Rightarrow, \vee, \wedge, \Leftrightarrow, S, W),$$

where the truth-tables of the first group of connectives, i.e. Kleene connectives, are the following:

[2] "The classical two-valued logic as an instrument of deduction, however, presupposes that neutral propositions are treated as if they were true, and inexact predicates as if they were exact.", Körner[3]

α	$\sim\alpha$
f	t
u	u
t	f

$\Rightarrow$	f	u	t
f	t	t	t
u	u	u	t
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	t
t	t	t	t

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

$\Leftrightarrow$	f	u	t
f	t	u	f
u	u	u	u
t	f	u	t

while the unary operations corresponding to idealisation connectives are described through Kleene-like tables as below.

α	$S\alpha$
f	f
u	t
t	t

α	$W\alpha$
f	f
u	f
t	t

A relation $\vdash_{K_3}$ is said to be a *matrix empirical inference* of K_3 provided that for any $X \subseteq For, \alpha \in For$

$$X \vdash_{K_3} \alpha \text{ iff for every } h \in Hom(L, A)(h(\alpha) = t \text{ if } \ h(X) \subseteq \{u, t\}).$$

Obviously, $\vdash_{K_3}$ is a q-consequence relation determined by the q-matrix

$$K_3^0 = (\underline{A}_3, \{u, t\}, \{t\})$$

The intuition behind its definition is that a conclusion α may be inferred empirically from a set of premises X, when for any interpretation if all elements of X are undetermined or true then α is true.

Now, consider the matrix

$$K_3 = (\underline{A}_3, \{t\})$$

and its consequence relation $\models^3$. Note, that S and W are two-valued and, therefore, we get

(••) $$X \vdash_{K^3} \alpha \text{ iff } S(X) \models^3 W(\alpha),$$

and, since $\models^3$ is finitary,

(i) $S(X_f) \models^3 W(\alpha)$ for some $X_f \subseteq X$.

In turn, since the conjunction on the set $\{f, t\}$ behaves classically, we may proceed as in Section 3. Thus, for a finite set of formulas $X_f = \{\beta_1, \beta_2, ..., \beta_s\}$, we take the conjunction $\bigwedge S(X_f)$ of all its formulas, i.e. the formula $S(\beta_1) \wedge S(\beta_2) \wedge ... \wedge S(\beta_s)$.

Then, $\models^3$ has the deduction theorem with respect to $\rightarrow$ and $\wedge$ and we may replace (iii) with

(ii) for some finite $X_f \subseteq X$, $\emptyset \models^3 \bigwedge S(X_f) \rightarrow W(\alpha)$

or, by

(iii) for some finite $X_f \subseteq X$, $(\bigwedge S(X_f) \rightarrow W(\alpha)) \in E(K_3)$.

The equivalence (i) $\equiv$ (iii) and (••) yield that $X \vdash_{K_3} \alpha$ is equivalent to (iii). Alas, we obtain a description of the q-consequence operation $Qn_{K_3^0}$ in terms of the content of the matrix K_3 or, more precisely, by a set of its selected tautologies:

$$Qn_{K_3^0}(X) = \{\alpha : \text{ for some finite } X_f \subseteq X, (\bigwedge S(X_f) \rightarrow W(\alpha)) \in E(K_3).$$

5. Concluding remarks

The representation of linear q-matrix inferences as "partial" consequence operations is highly intuitive. Moreover, it was succesfully applied in [4] as a tool enabling the uneasy step towards logical many-valuedness for any finite natural $n \geq 4$.

Getting a description of non-linear q-logics in terms of partial consequence operations seems to be a hard task. I conjecture, nonetheless, that a similar representation is possible for at least some structural q-consequence operations with no linear characteristics. The experience gained here suggests that worth trying are some classes of finitary q-consequences that have "regular" deductive properties, expressed in terms of set of connectives, sufficient for the description of sets of referential values of sentences and, thus, permitting to separate appropriate sets of formulas.

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