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FREGEAN AXIOM AND MANY-VALUEDNESS

Abstract

The Fregean Axiom, FA is an assumption that logical values and denotations of sentences are the same in number. Since the set of logical values consists of two elements: the true and the false, FA reduces the set of possible denotations of sentences to two, as well.

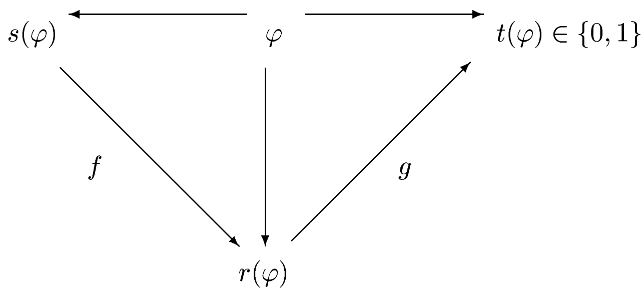
We aim at applying more logical values within the original Fregean Scheme. The proposal may ultimately be related to any finite number of values. However, in the present paper we will concentrate on the case of three-element set of values only. The most important reason for doing so is that this case may firmly be anchored in the the paradigm of inference relation of a q-consequence, see [2], [3]. The latter was constructed to formalize a reasoning which from non-rejected premisses leads to the accepted conclusions. A corresponding structural q-consequence operation felt apart from the Tarski paradigm of consequence and, finally appeared inferentially three-valued, thus violating the Suszko's thesis on logical two-valuedness, see [2].

The three-valued version of FA is first applied to logic algebras and matrices, and then to inference calculi defined or characterized by their use. The main result says that an inferential calculus (L, W) satisfies the three-valued version of FA if and if it is exclusively complete with respect to a class of not-reducible three-element matrices.

We believe that the proposal opens further possibilities for exploration of classes of models of logically three-valued inferential calculi having such characteristics.

1. Fregean semantic scheme as depicted by Suszko

The famous semantic scheme of Frege described in "Über Sinn and Bedeutung" may be, cf. [4], depicted as follows:



In the diagram, 1 and 0 represent the truth and falsity of sentences, respectively, φ is a name or a sentence, $r(\varphi)$ is the so-called referent (a denotation) of φ (i.e. what is given by φ) and $s(\varphi)$ the sense of φ (or the way $r(\varphi)$ is given by φ. If φ is a sentence, then $t(\varphi)$ is a logical (truth) value of φ, i.e. $t(\varphi) \in \{0,1\}$. These assignments are related as folllows:

(1) $s(\varphi) \neq s(\psi)$ whenever $r(\varphi) \neq r(\psi)$.

and, for sentences,

(2) $r(\varphi) \neq r(\psi)$ whenever $t(\varphi) \neq t(\psi)$.

The diagram commutes: i.e. given s, and t, there exist functions f and g such that

(3) $f(s(\varphi)) = r(\psi)$, and $g(r(\varphi)) = t(\psi)$.

Using the terminology elaborated by Suszko in his earlier writings, we may say that $r(\varphi)$ is the object denoted by φ provided that φ is a name, and a situation described by φ provided that φ is a sentence. In the end, we say that $s(\varphi)$ is a proposition expressed by φ.

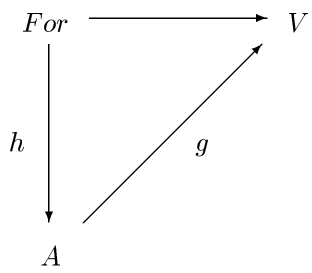
Fregean Axiom is an assumption that there are only two situations described by sentences. It amounts to

(FA) $t(\varphi) = t(\psi) \Rightarrow r(\varphi) = r(\psi)$.

An analysis of (FA) led Suszko to the construction of the so-called non-fregean logics with a non-truth functional identity connective, see [4].

2. FA for sentential calculi

Let For be a set of formulas of a sentential language L, and A a universe of an interpretation algebra A, $V = \{0,1\}$ the set of logical values, $h \in Hom(L, A)$ where $Hom(L, A)$ is the class of homomorphisms of L into For, and v a logical valuation, $v : For \to \{0,1\}$. Then, the following diagram is a counterpart of the semantic scheme of Frege:



Following the original idea, we assume that the present diagram commutes: given v, there is a function g such that for $\alpha \in For$

(4) $g(h(\alpha)) = v(\alpha)$.

As before, see (2), we also assume that

(5) $h(\alpha) \neq h(\beta)$, whenever $v(\alpha) \neq v(\beta)$.

The Fregean reference reducing condition applied to an algebra A is then a converse of (5) i.e. $v(\alpha) = v(\beta) \Rightarrow h(\alpha) = h(\beta)$. And, ultimately, the following equivalence is the Fregean Axiom for logical algebras;

(FA'_A) $v(\alpha) = v(\beta) \Leftrightarrow h(\alpha) = h(\beta)$

The whole framework easily extends onto logical matrices, which simply are interpretation algebras A equipped with a distinguished subset of elements corresponding to propositions of a specified kind. A matrix for a language L is a pair

$$M = (A, D),$$

with A being an algebra similar to L and $D \subseteq A$, a subset of A. Elements of D are *designated* of M.

The relation $\models_M \subseteq 2^{For} \times For$ is a *matrix consequence* of M provided that for any $X \subseteq For,\ \alpha \in For$

$$X \models_M \alpha \text{ iff for every } h \in Hom(L, A)(h\alpha \in D \text{ whenever } hX \subseteq D).$$

$\models_M$ is a natural generalization of the classical consequence and the set $E(M) = \{\alpha : \emptyset \models_M \alpha\}$ usually called the content of M is a counterpart of the set of tautologies. In the terminology just adopted, the classical matrix has the form

$$M_2 = (\{0,1\}, \neg, \rightarrow, \vee, \wedge, \equiv, \{1\}),$$

the classical consequence relation reads:

$$X \models_2 \alpha \text{ iff for every } h \in Hom(L, A_2)\ (h\alpha = 1 \text{ if } hX \subseteq \{1\}).$$

and the set of tautologies $TAUT$ is the content of M_2: $TAUT = E(M_2) = \{\alpha : \emptyset \models_2 \alpha\}$.

2.1. The Fregean Axiom FA_M is exclusively satisfied by two-element matrices or by the classes of two-element matrices.

2.2. Example. Consider the matrix standard two element matrix of the classical logic M_2 and its dual:

$$M_2 = (\{0,1\},\ \neg,\ \rightarrow,\ \wedge,\ \vee,\ \leftrightarrow,\ \{1\})$$

$$dM_2 = (\{0,1\},\ \neg,\ \rightarrow,\ \wedge,\ \vee,\ \leftrightarrow,\ \{0\})\ ,$$

where $\neg,\ \rightarrow,\ \wedge,\ \vee,\ \leftrightarrow$ are defined through the standard truth tables. These matrices, as well as the class $\{M_2,\ dM_2\}$ consisting of the both matrices satisfy FA_M.

Now, in order to adapt the previous decriptions to the classes of matrices we introduce a useful notion of a characteristics of a matrix and, ultimately a characteristics of a class of matrices. By a matrix characteristics, we understand the triple

$$< h,\ g,\ g \circ h >,$$

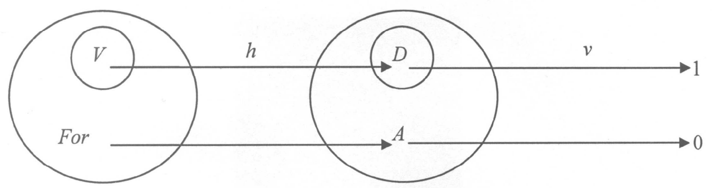
where h and g are as above and $g \circ h(\alpha) = g(h(\alpha))$. Where $\mathbf{K} = \{K_i : i \in I\}$ is a class of similar matrices, its characteristics is

$$\{< h_i,\ g_i,\ g_i \circ h_i > :\ i \in I\},$$

Then, the Fregean Axiom may be related to any sentential calculus $(L,\ C)$, where C is a structural consequence operation on L. Accordingly, $C = Cn_{\mathbf{K}}$ for some $\mathbf{K} = \{K_i : i \in I\}$, cf. [6]. FA ammounts then to the condition that for any $i \in I$ g_i is 1 - 1.

3. Valuations and logical three-valuedness

Suszko [1] stated that any matrix consequence, and therefore every structural consequence relation, may be described by means of 0 - 1 valuations and thus that every logic is logically two-valued. The idea that shifted logical values over the set of matrix values refers to the division of matrix universe into two subsets of *designated* and *undesignated* elements and use the characteristic functions of the set of designated elements D as *logical valuations*. The relation between homomorphic interpretations and logical valuations looks like



i.e. with every $h \in Hom(L, A)$ a function v_h: $For \to \{0, 1\}$ is associated such that $v_h(\alpha) = 1$ if whenever $h(\alpha) \in D$, and $v_h(\alpha) = 0$, otherwise.

The question whether many-valuedness of that kind is possible at all, led to the next mode of logical many-valuedness, more precisely the three-valuedness, which results from a natural generalization of Tarski's concept of consequence. The generalization described in [6] reflects the idea that the rejection and acceptance need not be complementary. The appropriate modelling relies on the so-called q-matrices with two distinguished sets of elements. A *q-matrix* for a given sentential language is a triple

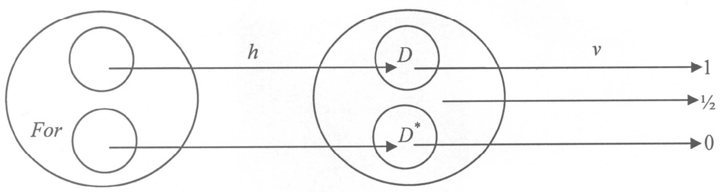
$$M^* = (A, D^*,\ D),$$

where D^*, D are disjoint subsets of A interpreted as rejected and accepted elements, respectively. A relation $\vdash_{M^*}$ is said to be a *matrix q-consequence of* M^* provided that for any $X \subseteq For, \alpha \in For$

$$X \vdash_{M^*} \alpha \text{ iff for every } h \in Hom(L, A)(h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset).$$

The relation of q-consequence was designed as a formal counterpart of reasoning admitting rules of inference which from non-rejected assumptions lead to accepted conclusions. The q-concepts coincide with usual concepts of matrix and consequence only if $D^* \cup D = A$, i.e. when the sets D^* and D are complementary. Then, the set of rejected elements coincides with the set of non-accepted elements.

In order to apply the Suszko's approach now we can hardly stay with 0 - 1 valuations, since in general it is possible that the third value for valuation is necessary - this is the case when D^* and D are not complementary. Accidently, the relation between homomorphic interpretations and trivalent logical valuations looks like



Now, with every $h \in Hom(L, A)$ a function $v_h : For \to \{0, 1/2, 1\}$ is associated such that $v_h(\alpha) = 1$ if $h(\alpha) \in D$, $v_h(\alpha) = 0$ if $h(\alpha) \in D^*$ and $v_h(\alpha) = 1/2$ whenever $h(\alpha) \in A - (D \cup D^*)$.

4. FA for inferentially three-valued logics

Due to Lindenbaum-Wójcicki's representation, cf. [4], [8], any structural q-consequence there is a class of q-matrices strongly adequate to W. Accordingly, W, in general, has not bivalent semantic characterization., since there are no-bivalent q-consequence operations. To any such q-matrix one may ascribe a three-valued characteristics

$$\{< h,\ g_i,\ g_i \circ h\ >:\ i \in I\},$$

where $v_i = g_i \circ h$.

The three-valuedness above is referential since $W(\emptyset) = C(\emptyset)$, where C is the consequence operation generated by he class of matrices received from M^* by cancelling all sets of rejected elements form the q-matrices of M^*.

That suggests a need of extending the Fregean scheme and the Fregean Axiom onto the case of three logical values. Thus, for a q-matrix $M^*(\mathrm{FA}_3)$ has the form

$$(\mathrm{FA}_3) \quad g_i \circ h(\alpha) = \ g_i \circ h(\beta) \Rightarrow h(\alpha) = h(\beta).$$

In turn, a class of q-matrices satisfies (FA_3) whenever all its matrices do.

4.1. (L, W) satisfies (FA_3) if and only if there is a class of q-matrices $\mathbf{M}$ generating W such, that every $M_i \in \mathbf{M}$ has three elements, and g_i is a 1 - 1 mapping from the matrix universe A and the set V.

4.2. Example. Consider M_3^*, its dual dM_3^* and $\{M_3^*, dM_3^*\}$:

$$M_3^* = \ (\{0, {}^1\!/_2, 1\},\ \sim,\ \rightarrow,\ \wedge,\ \vee,\ \leftrightarrow,\ \{0\},\ \{1\}\ )$$

$$dM_3^* = (\{0, {}^1\!/_2, 1\},\ \sim,\ \rightarrow,\ \wedge,\ \vee,\ \leftrightarrow,\ \{1\},\ \{0\})\ ,$$

where $\sim, \rightarrow, \wedge, \vee, \leftrightarrow$ are defined through the the Łukasiewicz three-valued matrices, with the connectives, which may be expressed by algebraic formulas: $\sim x = 1 - x$, $x \rightarrow y = min\{1, 1 - x + y\}$, $x \wedge y = max\{x, y\}$, $x \vee y = min\{x, y\}$ and $x \leftrightarrow y = 1 - |x - y|$. The matrices M_3^*, dM_3^*, the class $\{M_3^*, dM_3^*\}$ satisfy (FA_3). Accordingly, the corresponding sentential calculi also satisfy the Fregean Axiom in its three-valued version.

Final Remark. For matrices satisfying either version of the Fregean Axiom, (FA) or (FA_3), referential assignements (i.e. homomorpisms) and logical valuations coincide.

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