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Grzegorz Malinowski

Modes of many-valuedness*

Since the actual introduction of the third logical value next to truth and falsity by £ukasiewicz in 1918, several constructions of many-valued logic were proposed. The history of the topic already became rich and interesting. On the other hand, as is widely known, the problem of interpretation of many-valuedness is still among *vexed* questions of contemporary logic

The aim of the present paper is to put a stone into the discussion of the very nature of many-valued logic. First, in Section 1, the standard Rosser-Turquette sentential logics are presented as non-Fregean systems. In Section 2 a natural inference relation for Kleene strong three-valued logic is constructed and discussed. Since the author is of the opinion that the sentential (or, propositional) level decides whether a logic is many-valued or not, cf. Malinowski [1993], the following consideration is limited to the 0-order case.

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1. Syntax and semantic environment

Sentential languages are algebras constructed on a denumerable set of sentential variables, $Var = \{p, q, r, ...\}$ and a finite set $F = \{F_1, ... , F_m\}$ of sentential connectives. With each connective F_i there is associated natural number $a(F_i)$ that describes its syntactic category (i.e. number of arguments); to avoid empty cases we usually assume that at least one of the connectives is not trivial so that $a(F_i) \neq 0$. Then the set of formulae, *For*, is defined inductively, putting

(i) $Var \subseteq For$

(ii) For any $F_i \in$ F such that $a(F_i) = k$, $F_i(\alpha_1,..., \alpha_k) \in For$, whenever $\alpha_1,..., \alpha_k \in For$.

Thus, each sentential language is identified with an algebra of formulae

$$L = (For, F_1, ... , F_m) .$$

The usual rules of interpretation of formal language established already by Frege [1892], which require that to each formula exactly one semantic correlate is associated and that two formulae are interchangeable in any sentential context whenever they have the

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same correlate associated together imply, see Suszko [1957], that any interpretation structure for L

$$\mathcal{A} = (A, f_1, \ldots, f_m)$$

is an algebra similar to L. Moreover, any mapping s: $Var \rightarrow A$, may be extended uniquely to the homomorphism $h_s : L \rightarrow \mathcal{A}$, $h_s \in \mathrm{Hom}(L,\mathcal{A})$, and therefore, L is an absolutely free algebra in its similarity class. The last property simply means that the correlate (a value) of a compound formula may be calculated from correlates associated to sentential variables using description of the connectives.

Interpretation structures equipped with a distinguished subset of the set of semantic correlates corresponding to propositions of a specified kind (e.g. true sentences) are called logical *matrices* . More specifically, a pair

$$M = (\mathcal{A}, D),$$

with A being an algebra similar to a language L and $D \subseteq A$ a non-empty subset of the universe of $\mathcal{A}$, will be referred to as a matrix for L. Elements of D will be called *designated* elements of M.

With each matrix M for L there is associated a set of formulae which take designated values only:

$$E(M) = \{ \alpha \in \mathrm{For} : h\alpha \in D \text{ for any } h \in \mathrm{Hom}(L,\mathcal{A}) \}$$

called its *content*. In turn, a relation $\models_M$ is said to be a matrix consequence of M provided that for any $X \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$

$$X \models_M \alpha \text{ if and only if for every } h \in \mathrm{Hom}(L,\mathcal{A}) \; (h\alpha \in D \text{ whenever } hX \subseteq D).$$

The content of a given matrix is a counterpart of the set of tautologies and the entailment relation $\models_M$ is a natural generalization of the classical consequence.

2. Referential interpretation of standard logics

It is well known fact that the truth-functionality principle stating that the truth or falsity of every compound sentence depends only on logical values of its compounds constitutes the base of matrix description of the classical sentential logic.

The most natural and straightforward step beyond the two-valued logic is the introduction of more logical values, rejecting simultaneously the principle of bivalence. Such constructions are based on the logic matrices having more than two correlates. One easily remarks that the interpretation principle of sentential languages stated in Section 1 is a generalisation of the truth-functionality principle mentioned above. The link in question is somehow manifested also by the fact that most of the many-valued constructions are 'conservative' extensions of the classical logic matrices.

Below, we examine the Rosser and Turquette [1952] standard conditions which determine some finitely-valued sentential logics which resemble the CPC. Our purpose is to recover the relations between referential structure of RT matrices and classical core of the construction.

2.1. Standard conditions. Where $n \geq 2$ is a natural number. Assume that

$$E_n = \{0, {}^{1}/_{n-1}, {}^{2}/_{n-1}, \ldots ,1\}$$

is a set of possible semantic correlates of sentences. Let for a k, such that $1 < k \leq n-1$,

$$D_k = \{{}^{n-k+1}/_{n-1}, \ldots ,1\}$$

be a subset of E_n;. the elements of D_k will be referred to as *distinguished* (or, *designated*). Now, consider an algebra U_n with operations $f_1, f_2, \ldots , f_m$ having E_n as the universe:

$$U_n = (E_n , f_1, f_2, \ldots , f_m) .$$

Our consideration of the Rosser-Turquette many-valuedness will be focused on matrices

$$M_{n,k} = (U_n , D_k)$$

for an appropriate sentential language.[1]

Let $\neg, \rightarrow, \vee, \wedge, \leftrightarrow$ be (primitive or definable) functions of a given matrix $M_{n,k}$ corresponding to the connectives of negation, implication, disjunction, conjunction and equivalence, respectively. If for any $x, y \in E_n$

$\neg x \in D_k$ if and only if $x \notin D_k$
$x \rightarrow y \notin D_k$ if and only if $x \in D_k$ and $y \notin D_k$
$x \vee y \notin D_k$ if and only if $x \notin D_k$ and $y \notin D_k$
$x \wedge y \in D_k$ if and only if $x \in D_k$ and $y \in D_k$
$x \leftrightarrow y \in D_k$ if and only if either $x, y \in D_k$ or $x, y \notin D_k$

we will say that the respective connectives in the language satisfy the *standard conditions*, cf. Rosser, Turquette [1952]. In the sequel any matrix of the form

$$K_{n,k} = (E_n , \neg, \rightarrow, \vee, \wedge, \leftrightarrow , D_k)$$

having only standard connectives as functions is referred to as *Q-standard*.

It is easy to remark that each Q-standard matrix $K_{n,k}$ is epimorphic to the classical matrix $M_2 = (\{0,1\}, \neg, \rightarrow, \vee, \wedge, \leftrightarrow, \{1\})$ with functions described by truth tables. The required epimorphism of the two matrices is established by the mapping $e : E_n \rightarrow \{0,1\}$ defined as follows:

$$e(x) = \begin{cases} 1 & \text{when } x \in D_k \\ 0 & \text{when } x \notin D_k . \end{cases}$$

Now, we consider the family of one-argument connectives $J = \{ j_0, j_1, \ldots , j_{n-1} \}$ of U_n which satisfy the following *standard* condition:

[1] Originally, Rosser and Turquette used the numbers 1,2, ... ,k, ... , n to denote the elements of the matrices. The „ordering” was also reversed: 1 corresponded to the „truth”, while n was the falsity. The values 1, ... , k were designated.

$j_i(x) \in D_k$ if and only if $x = {}^{i}/_{n-1}$.

One quickly realises that these Rosser and Turquette connectives may be seen as a kind of „characteristic functions” identifying the respective semantic correlates. Thus, one might define an extension of *e* putting

$$e(j_i(x)) = \begin{cases} 1 & \text{if } x = {}^{i}/_{n-1} \\ 0 & \text{if } x \neq {}^{i}/_{n-1} . \end{cases}$$

Obviously, the presence of these connectives in the matrix lifts us out of the usual classical logic in this sense that e is not more (epi-)morphism into {0,1}. On the other hand, one may say that the extended logic may be described in terms of {0,1}- valuations; we postpone this question to the sequel.

2.2. Sentential identity. The classical sentential logic has a distinguished semantics provided by the two-element matrix M_2. The very special property of this characterisation is that the behaviour of the equivalence connective suits the Fregean condition stating that, from the point of view of the (classical logic), two sentences having the same logical values, describe the same i.e. have the same *referent* or, denotation. Intuitively speaking, we may say that the only attribute of sentence which counts for the classical logic is its truth-value. The truth table condition for the function of the equivalence expressed as above is as follows:

$x \leftrightarrow y \in \{1\}$ if and only if $x = y$. [2]

Let us note that the equality appearing on the right hand side of the formula is merely the *identity of the logical values* and not identity of sentences in any extended or deeper sense. The truth tables cover only a small part of the ontology of referents of sentences and in no reasonable sense they may tell anything about the contents of these linguistic entities.

For the purpose of avoiding this inconvenience, R.Suszko [1972] extended the classical logic introducing in the language of the classical logic a non truth-functional connective of identity, denoted henceforth as ≡. The intended meaning of the new connective is the best explained through models. i.e. matrices and it consists in expressing the fact that two sentences are identical, *modulo* given model, whenever their semantic correlates are identical. Relatively to a choice of the class of models one gets different kinds of sentential identity, which applies to diverse structures of universes semantic correlates including the distinctions between distinguished *situations*, i.e. those which obtain, and not distinguished, or negative. The weakest logic of sentential identity SCI, the Sentential Calculus of Identity, may be characterised semantically by the use of SCI-models, cf. Bloom [1971]. Actually, an SCI-model is a (proper) matrix $M = (A , D)$, consisting of an algebra

$$A = (A , \neg, \rightarrow, \vee, \wedge, \leftrightarrow, \equiv)$$

such that for any $a,b \in A$

[2] $x \leftrightarrow y \in \{1\}$ means, obviously, that $x \leftrightarrow y = 1$.

$\neg a \in D$ if and only if $a \notin D$
$a \rightarrow b \notin D$ if and only if $a \in D$ and $b \notin D$
$a \vee b \notin D$ if and only if $a \notin D$ and $b \notin D$
$a \wedge b \in D$ if and only if $a \in D$ and $b \in D$
$a \leftrightarrow b \in D$ if and only if either $a, b \in D$ or $a, b \notin D$
$a \equiv b \in D$ if and only if $a = b$.

The *referentially* defined SCI consequence relation $|=_{SCI}$ is introduced as follows:

$X \ |=_{SCI} \alpha$ if and only if $X \ |=_{M} \alpha$ for every SCI-model *M* .

SCI admits a great divergence of models. Effectively, there are no limitations on either cardinality or the internal algebraic structure of an intended model. Since, however, each interpretation of the SCI language L = (For, $\neg$, $\rightarrow$, $\vee$, $\wedge$, $\leftrightarrow$, $\equiv$) is a homomorphism h of L into A we may easily associate a bivalent logical valuation t_h : For $\rightarrow \{0,1\}$ such that

$t_h(\alpha) = 1$ if and only if $h(\alpha) \in D$.

Then, obviously, t_h in each case is a usual valuation of the truth-functional connectives as described by the classical truth tables. As for the identity, we have

$t_h(\alpha \equiv \beta) = 1$ if and only if $h(\alpha) = h(\beta)$,

So, clearly, SCI may be characterised through bivalent *logical valuations*, as opposed to homomorphisms being referential assignments.

Sentential Calculus with Identity considered either as the set of theorems, i.e. SCI-tautologies, as well as the SCI-entailment is very poor. Thus, e.g. all tautologies containing the identity connective are, in a sense, trivial. On the other hand, SCI proved to be an extensive and powerful protologic. Among its axiomatic and non-axiomatic strengthenings one may find important systems including modal systems S4 and S5 and the intuitionistic logic, cf. Malinowski [1985]. Obviously, a strengthening in each particular case on the semantic side means limiting the class of admissible models. Thus e.g. to get SCI-like semantics for S4 and S5, one has to consider only some special Boolean models.

2.3. Rosser-Turquette's protologics. Each non-trivial matrix $M_{n,k}$ considered in Section 2.1 having as a reduct the *classical* matrix $K_{n,k}$ is a good candidate for an n-element SCI model.
$M_{n,k}$ will be such whenever a connective of identity is definable. Actually, this is the case for all matrices having in their structures the J's connectives. So, assume that an $M_{n,k}$ is such and for any $x,y \in E_n$ let us put

$$x \equiv y = (j_0(x) \leftrightarrow j_o(y)) \wedge (j_1(x) \leftrightarrow j_1(y)) \wedge ... \wedge (j_{n-1}(x) \leftrightarrow j_{n-1}(y)) .$$

Using properties of the connectives on the right side one may verify that

$x \equiv y \in D_k$ if and only if $x = y$, and, thus, that $\equiv$ satisfies the condition for the sentential identity. Ultimately, the matrix

$$M_{n,k}^{\equiv} = (E_n, \neg, \rightarrow, \vee, \wedge, \leftrightarrow, \equiv, D_k)$$

is an n-element SCI-model.

Referentially n-valued *protologics* defined through the matrices $M_{n,k}^{\equiv}$ play a similar role to that which the Sentential Calculus with Identity has for the class of its strengthenings - axiomatic, at least. Since they contain the j connectives, it is always possible to redefine their semantic so as to receive a two-valued description, using the mapping *e* defined in Section 2.1. Thus, these logics are also logically two-valued in this sense that independently from the choice of other connectives all logical properties can be, with more or less difficulty, set in terms of {0,1}-valuations. In the end, let us note that the formulas

$$j_0(\alpha) \vee j_1(\alpha) \vee ... \vee j_{n-1}(\alpha)$$
$$\neg (j_s(\alpha) \wedge j_t(\alpha)) \text{ for } s \neq t, \ s,t \in \{0, 1, ... , n\text{-}1\}$$

both belong to $E(M_{n,k}^{\equiv})$, i.e. are tautologies of the logic under consideration. This means that our constructions are referentially n-valued: the minimal models should have n different elements. These elements may possibly be linearly ordered, as Rosser and Turquette would wish, and the ordering might be reflected by an *ordering connective*. I claim that the area of investigation of the variety of n-valued protologics is as rich as the domain of investigation of SCI and its strengthenings. Leaving this problem aside we finish the query giving a concrete example of a logic falling in this category.

2.3.1. Example.(compare Malinowski [1977]). Given a finite $n \geq 2$, let us consider two primary connectives on the n-valued £ukasiewicz logic: the negation ~, and the implication $\Rightarrow$. The matrix functions corresponding to these connectives are defined on the set E_n as follows: for any $x,y \in E_n$

$$\sim x = 1 - x$$
$$x \Rightarrow y = \mathit{min} \{1, 1- x + y \},$$

where the operations on the right hand side are arithmetical. Assume that k = 2, i.e. that {1} is the one-element set of designated elements. Then

$$£_n = (E_n, \sim, \Rightarrow, \{1\})$$

is the n-valued matrix of £ukasiewicz.

Using £-negation and £-implication it is possible to define the connectives $\neg$, $\rightarrow$, $\vee$, $\wedge$, $\leftrightarrow$ satisfying the standard conditions with respect to $£_n$ or, should we say, with respect to the matrix $£_{n,2}$. For example, we may take

$$\neg x =_{df} x \Rightarrow_{n-2} \sim [(\sim x \Rightarrow x) \Rightarrow x]$$
$$x \rightarrow y =_{df} x \Rightarrow_{n-1} y$$
$$x \vee y =_{df} (x \Rightarrow y) \Rightarrow y$$
$$x \wedge y =_{df} \sim (\sim x \vee \sim y)$$
$$x \leftrightarrow y =_{df} (x \rightarrow y) \wedge (y \rightarrow x)$$

as definitions.[3] Further to this, the £-equivalence $\equiv$ defined by

$$x \equiv y =_{df} (x \Rightarrow y) \wedge (y \Rightarrow x)$$

is the identity connective in $£_n$ - it is easy to check that $(x \equiv y) = 1$ if and only if $x = y$. In the end, as already Rosser and Turquette showed, the j connectives $j_0, j_1, \ldots, j_{n-1}$ are definable in the £ukasiewicz matrix. Therefore, $£_{n,2}^{\equiv} = (E_n, \neg, \rightarrow, \vee, \wedge, \leftrightarrow, \equiv, \{1\})$ is an n-element SCI-model. On the other hand, the £-connectives $\sim$, $\Rightarrow$ are definable through the functions of $£_{n,2}^{\equiv}$:

$$\sim x =_{df} (x \equiv \neg (x \equiv x))$$
$$x \Rightarrow y =_{df} ((x \wedge y) \equiv x)$$
$$x \Rightarrow y =_{df} ((x \vee y) \equiv y) .$$

So, as an easy corollary, we get that the n-valued £ukasiewicz logic is *equivalent* to an n-valued Rosser-Turquette protologic.

3. Inferential approach to Kleene logic

Below, we present and analyse a distinguished three-valued construction of Kleene, which due to its motivation falls out of the Rosser-Turquette's framework. The main feature of Kleene's three-valued logic is that the „third" value *undefiniteness* (or, *neutrality*) is not an independent value. Accordingly, it may be turned into either of two classical values, i.e. truth or falsity. In this case we also face the situation that no formula is a „tautology" and therefore considering an inference, or entailment relation is a matter of particular importance. Following some ideas by Körner [1966] we show that an unorthodox conception of inference relation different from the consequence may serve as an inferential base for Kleene's logic.

3.1. Kleene matrix and logic. The original three-valued logic by Kleene [1938] has an epistemological motivation. The principal Kleene's inspiration came from the studies of the foundations of mathematics, especially the problem of algorithms, cf. also Kleene [1952]. The third logical value was designed to mark indeterminacy of some proposition at a certain stage of scientific investigation. Consequently, its assumed status is somewhat different from the classical values of truth and falsity.

The starting point of Kleene's construction consists in considering the „third" category of propositions, such as whose logical value of truth or falsity is not essential for actual consideration, undefined or undetermined by means of accessible algorithms. Besides the classical values of truth (t) and falsity (f) he introduces the value of *undefiniteness* (u). The connectives: $\sim$ (negation), $\Rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction), and $\Leftrightarrow$ (equivalence) are characterised through the following truth tables:[4]

[3] $x \Rightarrow_{n-1} y$ replaces the chain $x \Rightarrow (x \Rightarrow \ldots (x \Rightarrow y) \ldots)$ with n-1 antecendents. The „value" of this implication is $\min \{ 1 , (n-1)(1 - x) + y \}$.

[4] Throughout the whole paper we are dealing only with the 1938's version of the Kleene logic describing the so-called *strong* connectives.

α	$\sim\alpha$
f	t
u	u
t	f

$\Rightarrow$	f	u	t
f	t	t	t
u	u	u	t
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	t
t	t	t	t

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

$\Leftrightarrow$	f	u	t
f	t	u	f
u	u	u	u
t	f	u	t

Kleene distinguishes the value of truth (t). The relevant feature of the *Kleene matrix*

$$K3 = (\{f,u,t\} , \sim , \Rightarrow , \vee , \wedge , \Leftrightarrow , \{t\})$$

with operations introduced by the above truth-tables is that it defines the *non-tautological* logic since any valuation which assigns the value u to each propositional variable sends any formula into {u}. Kleene is aware of discarding even such law of logic as the *law of identity* p $\Rightarrow$ p and its equivalential version p $\Leftrightarrow$ p. „Here „unknown" is a category into which we can regard any proposition as falling, whose value we either do not know or choose for the moment to disregard; and it does not then exclude the other two possibilities 'true' and 'false'." (Kleene [1952], p. 335). We may therefore conclude that the author of the construction treated the added logical value as apparent or as *pseudo-value*, distinct from the real truth-values, cf. Turquette [1963].

3.2. Experience and theory. The title of this paragraph is borrowed from Körner [1966] who gave the most accurate and compatible interpretation of Kleene's views of the connectives of K3. Körner introduced a non-standard abstract notion of an inexact class as an object characterised by a non-definite classifying procedure determined by a partial algorithm. He also showed that the algebra of inexact classes leads naturally to the three-valued logic K3. Basing on it Cleave [1974] worked out a model-theoretic framework for the logic of inexact predicates and, among others, he formulated the notion of logical consequence. Cleave's contribution is important with many respects, especially because of the analysis concerning the first-order inexact structures. From the point of view of the sentential logic, however, it is striking that Cleave construction adopts the idea that the Kleene values f, u, t are mutually comparable and linearly ordered: f $\leq$ u $\leq$ t. In the original Cleave notation -1 replaces f, 0 stands for u, and +1 for t. The concept of a special matrix consequence operation is determined by the Cleave's variant of *K3*,

$$C3 = (\{-1, 0, +1\} , \sim , \Rightarrow , \vee , \wedge , \Leftrightarrow , \{+1\}) ,$$

as follows: α is a *C3* consequence of the set of formulas X, X $\models_{C3} \alpha$, whenever for every valuation *v* of the language in *C3*, $\min \{v\beta : \beta \in X\} \leq v\alpha$. In view of the ordering of suggested above, we may say that $\models_{C3}$ is a case of *a degree of truth preserving* semantic consequence operation.

Given the prime significance associated by Kleene to the values f, u , t, one can hardly agree that they might be unconditionally comparable. The doubt concerns obviously the mutual comparability of u with the *classical* values f and t. One may try to solve this problem in several ways, possibly like in systems motivated by partiality in associating (classical) truth-values to sentences: supervaluations and the partial logic. Such proposals, however, would essentially affect the above framework of the consequence relation.

To prepare a ground for introducing an appropriate relation of inference, I would like to tackle some other important issues traced out by Körner[1966]. In his analysis of the general structure of scientific theories and their relation to experience Körner delineates two spheres of scientific activity: the sphere of experience and the sphere of theory. These are treated as complementary since the subject of scientific inquiry interprets the results of his empirical work in the theoretical discourse and, conversely, the concrete scientific theories impose further steps and influence the planning of subsequent activities. Körner's inquiry into the logico-mathematical frameworks of scientific theories tends him to claim that every deductive unification of a field of experience by classical logic is an idealisation or modification of empirical discourse. He even maintains that the empirical and the theoretical discourse are, by virtue of the fact that the latter is an idealisation of the former, logically disconnected. In what follows we will focus our attention on problems of *external* construction and evaluation of theoretical and empirical sentences following the line of Körner's thought.

We assume that empirical propositions may take three values: f, u, t interpreted as Kleene wished. Further to this, we assume that u is incomparable with f and t. Now, the compound empirical sentences are formed from (atomic) empirical sentences using the connectives $\sim, \Rightarrow, \vee, \wedge, \Leftrightarrow$ interpreted in accordance with the tables in Section 2.1. On their hand, the theoretical sentences are true or false and their logic is classical. Our first task in the sequel is to localise the procedure of idealisation as bridging the two kinds of sentences. Further, we show how a new paradigm affects foundations of entailment. In the end, we will define a natural inference relation in the empirical discourse.

3.3. Empirical inference. Departing from the idea of „neutrality"of sentences of the third category and accepting inexactness Körner [1966] tries to apply the classical logic to empirical discourse. To this aim, he proposes a special evaluation device leading to a kind of „modified two-valued logic" and, ultimately, to an instrument of evaluation validity of sentences and deduction. An interesting and natural consideration of the problem ends negatively: „The classical two-valued logic as an instrument of deduction, however, presupposes that neutral propositions are treated *as if* they were true, and inexact predicates *as if* they were exact. ..."[5] This obviously means that the use of the classical logic for drawing empirical conclusions from empirical premises is unsound.

Another possibility for passing from the empirical inexact and undefinite discourse to the „heaven" of theory is using *idealisation* procedures. In case of (inexact) classes this means replacement by their exact counterparts,[6] in case of sentences turning undefinite sentences into definite, i.e. true or false. We may imagine even that every instance of empirical inference has to be preceded by its proper procedure of an idealisation. Now, we think that the empirical inference defined on the language of Kleene three-valued sentential logic in the wording established in Section 1 should look like below.

A relation $|\!\!-\!\!-_{K3}$ is said to be a *matrix empirical inference* of *K3* provided that for any $X \subseteq For$, $\alpha \in For$

$$X \ |\!\!-\!\!-_{K3}\ \alpha \quad \text{if and only if for every } h \in Hom(L,A) \ (h\alpha = t \text{ whenever } hX \subseteq \{u,t\}) .$$

[5] Körner [1966], p.47.

[6] compare Cleave [1974]

An intuition behind this definition is that a conclusion α may be inferred empirically form a set of premises X, when for any interpretation it is the case that if all elements of X are not false then α is true. We argue that this is just what one may expect from an entailment compatible with two possible logical idealisations of undefinite empirical sentences: first, turning it into true and the other, into false sentence. The operators corresponding to these idealisations are presented through Kleene-like tables as follows:

α	Wα
f	f
u	f
t	t

α	Sα
f	f
u	t
t	t

The letters chosen for these connectives may be read as *Weak* and *Strong*. Let us note that the empirical inference just defined may be also interpreted as a relation which holds between a set of premises X and a conclusion α whenever α is true independently from accepted idealisation(s) of empirical sentences in X.

Here are examples of theorems establishing some essential properties of the empirical inference of Kleene sentential logic:

(i) If $X \;|\text{--}_{K3}\; \alpha$ and $X \;|\text{--}_{K3}\; \alpha \Rightarrow \beta$, then $X \;|\text{--}_{K3}\; \beta$,

(ii) $X \;|\text{--}_{K3}\; \alpha \Rightarrow \beta$ if and only if $X \cup \{\alpha\} \;|\text{--}_{K3}\; \beta$

(iii) $\alpha \;|\text{--}_{K3}\; S\alpha$ and $W\alpha \;|\text{--}_{K3}\; \alpha$.

The first property is an *inferential modus ponens* , the second a *conditional deduction theorem*. The inferences in (iii) state that from a sentence α its strong idealisation follows while the sentence α itself is inferred from its weak idealisation.

4. Conclusions and final remarks

In the paper two interpretations of many-valued logic have been presented. Here, we offer some further explanation and remarks on the understanding of these two modes of many-valuedness.

4.1. Referential many-valuedness. The classical logic is both referentially and logically two-valued. Due to the Fregean assumption [FA] stating that two sentences having equal truth-values are identical (i.e. they refer to the same) the two qualities coincide, cf. 2.2. The situation changes radically in non-classical logic since we break [FA] and differentiate the referential and the logical *valuedness*. Consequently, homomorphisms associating sentences with their possible semantic correlates (i.e. referents or situations) may be set against the logical valuations being zero-one-valued functions on *For*. The procedure forms a part of a broader semantical programme related to the conception of the so-called non-Fregean logics (Suszko [1972]). Suszko claims that each sentential logic, i.e.

a *structural* consequence relation, can be determined by a class of logical valuations and thus, it is *logically two-valued*, cf. Suszko [1977].

In Section 2, n-valued (n finite) standard many-valued logics are interpreted as systems or strengthenings of the Sentential Calculus of Identity. For any permissible choice of n and k standard Rosser-Turquette matrices prove to be n-element SCI models. This is due to the fact that all *usual* connectives are standard and that the identity connective is definable in terms of the conjunction, equivalence and j-connectives. Consequently, every such matrix may serve as a base for a class of n-valued logics. Taking into account a distinct position of primary systems defined through the original matrices we referred to them as to protologics.

The proposed description of standard Rosser-Turquette logics shows that the idea of referential many-valuedness is compatible with the conception of logical bivalence.

4.2. Inferential many-valuedness. The matrix empirical inference relation $|\!\!-\!\!-_{K3}$ defined in 3.3. appears to be a special case of the so-called *q-consequence* relation introduced and studied in Malinowski [1990]. The semantic characterisation of this relation is based on a division of the set of semantic correlates into the three parts. Thus, a *q-matrix* for a given sentential language is a triple

$$M^* = (A\ ,\ D^*\ ,\ D)\ ,$$

where D^*, D are disjoint subsets of A interpreted as rejected and designated elements of M^*, respectively. A relation $|\!\!-\!\!-_{M^*}$ is said to be a *matrix q-consequence of M** provided that for any $X \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$

$$X\ |\!\!-\!\!-_{M^*}\ \alpha \text{ if and only if for every } h \in \mathrm{Hom}(L,A)\ (h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset).$$

The relation of q-consequence was designed as a formal counterpart of reasoning admitting rules of inference which from non-rejected assumptions lead to accepted conclusions. The q-concepts coincide with usual concepts of matrix and consequence only if $D^* \cup D = A$, i.e. when the sets $D^*$and D are complementary. Then, the set of rejected elements coincides with the set of non-designated elements.

One may easily show, that for any q-matrix M^* for which $D^* \cup D \neq A$ no class TV of functions $t : \mathrm{For} \to \{0,1\}$ exists such that for all $X \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$, $X\ |\!\!-\!\!-_{M^*}\ \alpha$ iff for each $t \in TV$, ($t(X) \subseteq \{1\}$ implies $t\alpha = 1$). Thus, a proper q-consequence relation is not logically two-valued. One may, however, give a three-valued description of $|\!\!-\!\!-_{M^*}$ and thus it may be conceived as logically three-valued inference, cf. Malinowski [1994].

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Department of Logic,
University of Łódź,
e-mail: gregmal@.uni.lodz.pl