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INFERENTIAL EXTENSIONS OF ŁUKASIEWICZ MODAL LOGIC*

Two inferential extensions of the Łukasiewicz system of modal logic are propositional logics based on the so-called q-consequence operation introduced in [2]. The main feature of q-consequence is that its rules lead from non-rejected assumptions to accepted conclusions. We present q-consequence and introduce concepts of extensionality and intensionality indistinguishable within the Tarski paradigm of consequence. The q-consequence operations for **Ł**-modal system prove to be good experimental range for expressing unorthodox notions of extensionality and intensionality. They also permit to distinguish between the two "indistinguishable connectives of possibility", cf [1].

1. Łukasiewicz system of four-valued propositional logic **Ł** has as its purpose capturing the modal notions of possibility. Its algebra of logical values was a product of two Boolean algebras with implication and negation and one-argument operations of: *assertion* A (the first) and *verum* V (the second); i.e., $(\{0,1\}, \rightarrow, \neg, A)$ and $(\{0,1\}, \rightarrow, \neg, V)$, where $A(0) = 0, A(1) = 1$, and $V(0) = V(1) = 1$. The values were, primarily, the ordered pairs $(1,1), (1,0), (0,1)$ and $(0,0)$. The product had three operations: $\rightarrow$ (implication), $\neg$ (negation) and Δ (possibility), identified with the "cross" product of A and V. Łukasiewicz also considered the second "possibility" ∇ twin to Δ. He also simplified the notation and used 1 to stand for $(1,1), 2$ for $(1,0), 3$ for $(0,1)$ and 4 for $(0,0)$. The Łukasiewicz logic algebra in this notation has the form:

$$£ = (\{1,2,3,4\}, \rightarrow, \neg, \Delta, \nabla),$$

with operations defined by the following tables:

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$\to$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

	$\neg$
1	4
2	3
3	2
4	1

	Δ
1	1
2	1
3	3
4	3

	∇
1	1
2	2
3	1
4	2

.

The system **Ł** of modal logic was defined on the language $\mathcal{L} = (For, \to, \neg, \Delta, \nabla)$ as the set of all formulas taking for every valuation of $\mathcal{L}$ in £ the *distiguished value* 1, thus

Ł $= \{\alpha \in For :$ for every $h \in \mathrm{Hom}(\mathcal{L}, £), h(\alpha) = 1\}$.

2. Let $\mathcal{L} = (For, F_1, \ldots, F_m)$ and $\mathcal{M} = (M, f_1, \ldots, f_m)$ be a propositional language and an algebra similar to it, respectively. The semantic characterisation of the q-consequence relation defined on a given interpretation structure is based on a division of the set of semantic correlates into three parts. Thus, a q-matrix for $\mathcal{L}$ is a triple

$\mathrm{M}^* = (\mathcal{M}, D^*, D)$,

where D^* and D are disjoint subsets of $M, D^* \cap D = \emptyset$, interpreted as *rejected* and *accepted* elements of M^*. A relation $\vdash_{\mathrm{M}^*}$ is said to be a matrix q-consequence of M^* provided that for any $X \subseteq For, \alpha \in For$

$X \vdash_{\mathrm{M}^*} \alpha$ if and only if for every $h \in Hom(\mathcal{L}, \mathcal{M})(h\alpha \in D$ whenever $hX \cap D^* = \emptyset$.

The q-concepts coincide with usual concept of logical matrix and consequence only if $D^* \cup D = M$, i.e. when sets D^* and D are complementary. Then the set of rejected elements coincides with the set of non-designated elements; M^* and $\vdash_{\mathrm{M}^*}$ reduce to

$\mathrm{M} = (\mathcal{M}, D)$,

$X \vdash_{\mathrm{M}} \alpha$ if and only if for every $h \in \mathrm{Hom}(\mathcal{L}, \mathcal{M})(h\alpha \in D$ whenever $hX \subseteq D)$.

With any relation $\vdash_{\mathrm{M}^*}$ an operation $Wn_{\mathrm{M}^*} : \mathcal{P}(For) \to \mathcal{P}(For)$ is associated, such that for any set of formulas X

$Wn_{\mathrm{M}^*}(X) = \{\alpha \in For : X \vdash_{\mathrm{M}^*} \alpha\}$.

Similarly, for $\vdash_{\mathrm{M}}$, the associated consequence operation $Cn_{\mathrm{M}} : \mathcal{P}(For) \to \mathcal{P}(For)$ is

$$Cn_{\mathrm{M}}(X) = \{\alpha \in For : X \vdash_{\mathrm{M}} \alpha\}.$$

In the end, $Wn_{\mathrm{M}^*}(\emptyset) = Cn_{\mathrm{M}}(\emptyset)$, i.e. that the *sets of tautologies* of the two semantic structures coincide. Therefore, a system of formulas valid in a given matrix M may be extended to the *sentential calculus* $(\mathcal{L}, Cn_{\mathrm{M}})$ or to $(\mathcal{L}, Wn_{\mathrm{M}^*})$. The latter will be called the *inferential* since the *repetition rule* is not universally accepted in it, cf [2].

3. $W : \mathcal{P}(For) \to \mathcal{P}(For)$ is a *q-consequence* operation on $\mathcal{L}$ whenever for any $X, Y \subseteq For$

(W1) $X \subseteq Y$ implies $W(X) \subseteq W(Y)$,
(W2) $W(X \cup W(X)) = W(X)$.

If for every endomorphism $e \in End(\mathcal{L})$, i.e., substitution of $\mathcal{L}, eW(X) \subseteq W(eX), W$ is called *structural.*

The main feature of the q-consequence is that in general the *repetition rule* $(\{\alpha\}, \alpha : \alpha \in For)$ is not unlimitely valid. So, for some formulas α, β the condition

(i) $\alpha \in W(X)$ if and only if $\beta \in W(X)$

does not imply

(ii) $W(X, \alpha) = W(X, \beta)$.

The property just mentioned gives rise to the distinction between the two relations between formulas, one corresponding to (i) and the second to (ii). There are good reasons to understand the second one as "identity with respect to $W(X)$" and the first as "equivalence with respect to $W(X)$"; referring to the Tarski's theory of consequence.

Now, we put that in general terms. Hereafter, the symbols like $\varphi(\alpha/p)$ and $\varphi(\beta/p)$, where α, β and φ are formulas and p is a propositional variable, stand for the formulas resulting form φ by substituting the formula α or β, respectively, for all occurrences of p. Given a *q*-consequence W on $\mathcal{L}$, we define two binary relations: $=_W$ and $\approx_W$ on For, putting

(1) $\alpha =_W \beta$ if and only if $W(X, \varphi(\alpha/p)) = W(X, \varphi(\beta/p))$
for every $\alpha, \beta, \varphi \in For, X \subseteq For$ and $p \in Var$.

(2) $\alpha \approx_W \beta$ if and only if $\varphi(\alpha/p) \in W(X)$ iff $\varphi(\beta/p) \in W(X)$
for every $\alpha, \beta, \varphi \in For, X \subseteq For$ and $p \in Var$.

One may easily remark that in case when W is a consequence operation, i.e., when also $X \subseteq W(X)$ for every $X \subseteq For$ (or, equivalently, when the repetition rule is unlimitedly accepted), the two relations coincide.

4. Two objects are identical if and only if they share exactly the same properties. If we apply this *Leibniz dictum* for a *q*-consequence W, then $\alpha =_W \beta$ should imply for any φ, p and X, if $\varphi(\alpha/p) \in W(X)$ then $\varphi(\beta/p) \in W(X)$ and vice-versa. Thus, then also $\alpha \approx_W \beta$. Taking that into account we shall call W to be *extensional* whenever $=_W \subseteq \approx_W$.

The property of extensionality may naturally be adopted to propositional contexts and, ultimately, to connectives. Where $\varphi \in For$, formulas α and β will be referred to as (φ, W)-*equivalent*, $\alpha \ {}_{\varphi}{\approx}_W \ \beta$, provided that for any $X \subseteq For$ and $p \in Var : \varphi(\alpha/p) \in W(X)$ if and only if $\varphi(\beta/p) \in W(X)$. It is obvious that

4.1. $\approx_W = \bigcap\{\alpha \ {}_{\varphi}{\approx}_W \ \beta : \varphi \in For\}$.

4.2. *Each connective of extensional inference W is W-extensional.*

Let us now consider the following two q-matrices related to the Łukasiewicz system **Ł**:

(3) $\mathrm{M}£_{\Delta} = (£, \{3, 4\}, \{1\})$,

(4) $\mathrm{M}£_{\nabla} = (£, \{2, 4\}, \{1\})$.

The choice of the sets of rejected and accepted elements in $\mathrm{M}£_{\Delta}$ and in $\mathrm{M}£_{\nabla}$ and the whole idea of considering inferential extensions of the system of modal logic are in a way connected with Łukasiewicz attempts to discern between the two operators of possibility. Note, that in the first case rejected are those elements of the algebra of values, which Δ "sends to" not designated values (i.e. different from 1).

The *q*-matrices $\mathrm{M}£_{\Delta}$ and $\mathrm{M}£_{\nabla}$ define two inferential extensions of **Ł**-modal logic, i.e. the following inferential calculi:

(5) $(\mathcal{L}, W_\Delta)$ and $(\mathcal{L}, W_\nabla)$;

we put W_Δ for $Wn_{\mathrm{M}\pounds_\Delta}$, and W_∇ for $Wn_{\mathrm{M}\pounds_\nabla}$. Obviously, $W_\Delta(\emptyset) = W_\nabla(\emptyset) = \mathbf{Ł}$.

We show that W_Δ is an intensional q-consequence and all connectives but Δ are W_Δ-intensional. Thus in the calculus $(\mathcal{L}, W_\Delta)$ one possibility operator Δ is extensional (!) and the other, ∇ , intensional. Similarly, in $(\mathcal{L}, W_\nabla)$ all connectives but ∇ are intensional. These properties are of some interest from the point of view of Łukasiewicz's opinion concerning indistinguishability of two possibility connectives and seem to be relevant to some points undertaken by Simons in [3].

References

[1] J. Łukasiewicz, *A system of modal logic*, **The Journal of Computing Systems** 1 (1953), pp. 111–149.

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[3] P. Simons, Łukasiewicz, Meinong and many-valued logic, K. Szaniawski (ed.), **The Vienna Circle and the Lvov-Warsaw School**, Kluwer Academic Publishers, 1989, pp. 249–291.

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