Grzegorz Malinowski

MANY-VALUED REFERENTIAL MATRICES

The research program iniciated by the possible worlds semantics gave several constructions in which logical values of truth ad falsity are associated with points of reference and sentences. Since the pioneering work by Kripke several attempts have been made to get general rules of producing this kind of semantics.

An important step in this direction was made by Wójcicki, who in [5] defined matrices having functions from a set of indices into $\{0, 1\}$ as elements. The scope of referential semantics built then in the way accepted in the theory of logical calculi was limited to those structural consequence operations C for which the so-called C-equivalence and C-congruence coincide, i.e. for selfextensional logics. Referential matrices found applications and were subsequently generalised for the whole class of propositional logics, cf. [2]. The recent presentation in [6] shows how referential matrices are related to frames and possible worlds paradigm.

In the paper we define some natural many-valued extension of the concepts of referential matrix and semantics inspired by Rosser and Turquette's formalisation of a class of finite-valued propositional calculi in [3].

Given a propositional language $\underline{L} = (L, F_1, \ldots, F_k)$ and a non-empty set of reference points T let us call a ramified matrix $W = (\underline{A}_W, D_W)$ associated with $\underline{L}$ an *m-valued referential matrix for* $\underline{L}$ *defined relative to* T iff the following conditions are satisfied:

(W1) The set A_W of elements of the algebra $\underline{A}_W$ similar to $\underline{L}$ is a subset of $\{e_0, e_1, \ldots, e_{m-1}\}^T$, i.e. the elements of $\underline{A}_W$ are functions of the form r: $T \to \{e_0, e_1, \ldots, e_{m-1}\}$. In what follows we shall put $e_0 = 0$ and $e_{m-1} = 1$.

(W2) D_W is the family of all sets $D_t = \{r \in A_W : r(t) = 1\}$.

(W3) If $m \geq 3$, then in $\underline{A}_W$ there are definable $(m-2)$ no constant functions $E_1^o, \ldots, E_{m-2}^o$ such that

$$E_i^o(r)(t) = \begin{cases} 1 & \text{whenever } r(t) = e_i \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY 1. *2-valued referential matrices are defined by (W1) and (W2) only and thus they are just referential matrices in the sense of [4].*

Where W is a (ramified) matrix associated with L, Cn_W will be used to denote the matrix consequence operation determined by W in L. Similarly, for a class K of matrices for L, Cn_K will denote the consequence operation determined by K. Finally, by a propositional logic we mean any pair (L, C) where C is a structural consequence in L. If there is any, all other terminology comes from [6].

PROPOSITION 1. *Let W be an m-valued referential matrix for $\underline{L}$ defined relative to T. Then, for each $\alpha \in L$ and each $X \subseteq L$ the following two conditions are equivalent:*

(i) $\alpha \in Cn_W(X)$

(ii) For each valuation h in W, and each $t \in T, (h\alpha)(t) = 1$ whenever $(h\beta)(t) = 1$ for all $\beta \in X$.

PROOF. Obvious, cf. [5].

Where $m \geq 2$, each set K of m-valued referential matrices for $\underline{L}$ will be referred to as an *m-valued referential semantics* for $\underline{L}$ and then the logics $(\underline{L}, Cn_K)$ will be called *m-referential*.

Applying similar argument to that used for the proof of (A) Proposition on p. 379 in [6] we get the following

PROPOSITION 2. *A propositional logic $(\underline{L}, C)$ is m-referential for some $m \geq 2$ iff there exists an m-valued referential matrix W such that $C = Cn_W$.*

Our nearest task is to give conditions necessary and sufficient for a logic $(\underline{L}, C)$ to be m-referential for some $m \geq 3$.

Where $m \geq 3$ is a finite natural number, we shall say that a propositional logic $(\underline{L}, C)$ is *m-normal* iff in $\underline{L}$ there are definable $(m-2)$ unary connectives $E_1, \ldots, E_{m-2}$ such that for any $i, j \in \{1, \ldots, m-2\}$

(N0) $C(E_i(p)) \neq L$,
(N1) $C(\alpha, E_i(\alpha)) = L$,

(N2) $C(E_i(E_j(\alpha))) = L$,
(N3) $C(E_i(\alpha), E_j(\alpha)) = L$ whenever $i \neq j$

for each $p \in Var(L)$ and $\alpha \in L$.

Now, using the common notations we assume, for a given $(\underline{L}, C)$, $\sim_C$ and $\approx_C$ to be the relations on L defined by

(1) $\alpha \sim_C \beta$ iff $C(\alpha) = C(\beta)$,

(2) $\alpha \approx_C \beta$ iff $C(\varphi(\alpha/p)) = C(\varphi(\beta/p))$ for every $\varphi \in L$.

In turn, assuming that $(\underline{L}, C)$ is m-normal for some $m \geq 3$, define for each $i \in \{1, \ldots, m-2\}$ the relation $\sim_i$ on L in the following manner:

(3) $\alpha \sim_i \beta$ iff $C(E_i(\alpha)) = C(E_i(\beta))$.

Corolarry 2. *It is easy to see that $\sim_C, \sim_1, \ldots, \sim_{m-2}$ are equivalence relations on L, and (cf. e.g. [5]) that $\approx_C$ is a congruence of $\underline{L}$.*

Theorem. *A propositional logic $(\underline{L}, C)$ is m-referential $(m \geq 3)$ iff it is m-normal and $\approx_C \; = \; \sim_C \cap \sim_1 \ldots \cap \sim_{m-2}$.*

Proof. $(\rightarrow)$. Assume that $(\underline{L}, C)$ is m-referential for some $m \geq 3$. Then, from Proposition 2 it follows that there is an m-valued referential matrix W such that $C = Cn_W$. Where $E_1, \ldots, E_{m-2}$ is any selection of connectives corresponding to the functions $E_1^o, \ldots, E_{m-2}^o$ of $\underline{A}_W$, it is easy to check that the conditions (N1) – (N3) hold true. Thus, the logic $(\underline{L}, C)$ is m-normal. In turn, suppose that for some $\alpha, \beta \in L$

(*) $\alpha \sim_C \beta$ and $\alpha \sim_i \beta$ for all $i \in \{1, \ldots, m-2\}$.

Then, using our main assumption ($C = Cn_W$), (1) and (3) we get that for every valuation $h : \underline{L} \rightarrow \underline{A}_W$

(4) $(h\alpha)(t) = 1$ iff $(h\beta)(t) = 1$

and

(5) $(hE_i(\alpha))(t) = 1$ iff $(hE_i(\beta))(t) = 1$.

Clearly, (5) is equivalent to the condition: $E_i^o(h\alpha)(t) = 1$ iff $E_i^o(h\beta)(t) = 1$ and consequently

(6) $(h\alpha)(t) = e_i$ iff $(h\beta)(t) = e_i$

and therefore we finally obtain $h\alpha = h\beta$. This obviously yields $C(\varphi(\alpha/p)) = C(\varphi(\beta/p))$ for every φ. So, $\alpha \approx_C \beta$. Thus, we have proved that $\sim_C \cap \sim_1 \ldots \cap \sim_{m-2} \subseteq \approx_C$ and since the opposite inclusion is trivially valid, we get $\approx_C = \sim_C \cap \sim_1 \cap \ldots \cap \sim_{m-2}$.

($\leftarrow$). Let Δ be an arbitrary closure base for C consisting of C-consistent sets (for instance one can put $\Delta = \{C(X) : X \subseteq L$ and $C(X) \neq L\}$. The family Δ will serve as the set of indices of the m-valued referential algebra $\underline{A}_W$ we are going to define.

For each $\alpha \in L$, let α^Δ be the function from Δ into $\{0, e_1, \ldots, e_{m-2}, 1\}$ such that

(R1) $\alpha^\Delta(X) = 1$ iff $\alpha \in X$
(R2) $\alpha^\Delta(X) = e_i$ $(i \in \{1, \ldots, m-2\})$ iff $\alpha \notin X$ and $E_i(\alpha) \in X$
(R3) $\alpha^\Delta(X) = 0$ iff $\alpha \notin X$ and $E_i(\alpha) \notin X$ for all $i \in \{1, \ldots, m-2\}$.

Notice that the definition of α^Δ is correct: first, if for some X (Δ consists of C-consistent systems only) $\alpha \in X$, then according to (N1), $E_i(\alpha) \notin X$ for all $i \in \{1, \ldots, m-2\}$; and, if for some i $E_i(\alpha) \in X$, then by (N3) $E_j(\alpha) \notin X$ for all $j \neq i$. The set $A_W = \{\alpha^\Delta : \alpha \in L\}$ will be the carrier of our $\underline{A}_W$.

In turn, let F_S be an arbitrary k-argument connective of the language $\underline{L}$. Then, we shall assign to F_S the operation f_S on A_W defined as follows:

(7) $f_S(\alpha_1^\Delta, \ldots, \alpha_k^\Delta) = (F_S(\alpha_1, \ldots, \alpha_k))^\Delta$.

To prove that f_S is well defined we have to show that for any two sequences of formulas: $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ the condition

(8) $\alpha_1^\Delta = \beta_1^\Delta, \ldots, \alpha_k^\Delta = \beta_k^\Delta$

implies

(9) $(F_S(\alpha_1, \ldots, \alpha_k))^\Delta = (F_S(\beta_1, \ldots, \beta_k))^\Delta$.

Under (R1) – (R3) (8) is equivalent to

(8') For any $X \in \Delta$ and any $j \in \{1, \ldots, k\}, i \in \{1, \ldots, m-2\}$
$\alpha_j \in X$ iff $\beta_j \in X$ and $E_i(\alpha_j) \in X$ iff $E_i(\beta_j) \in X$,

and since Δ is a base of C we then have $\alpha_j \sim_C \beta_j, \alpha_j \sim_1 \beta_j, \ldots, \alpha_j \sim_{m-2} \beta_j$ what together with our assumption implies

(10) $\alpha_j \approx_C \beta_j$

(11) $F_S(\alpha_1, \ldots, \alpha_k) \approx_C F_S(\beta_1, \ldots, \beta_k)$

which obviously yields (9).

Thus, we have defined an m-valued algebra $\underline{A}_W$ referential with respect to Δ. Subsequently, we put $W = (\underline{A}_W, D_W)$, where D_W is given as in (W2). What we have still to prove is that the matrix W is normal, i.e. that it posseses the property mentioned in (W3).

For every $i \in \{1, \ldots, m-2\}$ let us define the function E_i^o in $\underline{A}_W$ as follows:

(12) $E_i^o(\alpha^\Delta)(X) = (E_i(\alpha))^\Delta(X)$.

Then:

1^o. If $\alpha^\Delta(X) = e_i$ for some $X \in \Delta$ and $i \in \{1, \ldots, m-2\}$, then from (R2) we get $E_i(\alpha) \in X$ and therefore by (R1) $(E_i(\alpha))^\Delta(X) = 1$. In turn, for every $j \neq i$ $E_j(\alpha) \notin X$ (by (N3)) and, by (N2) for any $k \in \{1, \ldots, m-2\}$ $E_k(E_j(\alpha)) \notin X$ what by (R3) yields $(E_j(\alpha))^\Delta(X) = 0$.

2^o. If $\alpha^\Delta(X) = 1$, then $\alpha \in X$ and due to (N1) $E_i(\alpha) \notin X$ for any $i \in \{1, \ldots, m-2\}$. Subsequently, (N2) yields $E_j(E_i(\alpha)) \notin X$ and therefore by (R3) we obtain $(E_i(\alpha))^\Delta(X) = 0$.

3^o. If $\alpha^\Delta(X) = 0$, then by similar reasoning, using (R3) and (N2), we can show that for every $i \in \{1, \ldots, m-2\}$ $(E_i(\alpha))^\Delta(X) = 0$.

So, the constructed m-valued matrix W is normal (E_i^o's are defined as in (12)). And, for ending the proof of the theorem, it remains to show that $Cn_W = C$. This, however, goes quite similarily to the corresponding proof for the case of the two-valued referential semantics, compare 5.6.8 Lemma in [6], and therefore it will be omitted.

Corollary 3. *$(\underline{L}, C)$ is 2-referential, i.e. referential in the sense of [5], if and only if $\approx_C = \sim_C$. Such logics are in [5] and [6] called selfextensional.*

Examples.

I. (Given a finite $n \geq 3$) *n-valued propositional calculus of Łukasiewicz* $Ł_n = (\underline{L}, C_n)$ is *n-referential.*

$C_n = Cn_{M_n}$ where $M_n = (\{0, 1/n\text{-}1, \ldots, 1\}, \rightarrow, \vee, \wedge, \neg, \{1\})$ is the

n-valued Łukasiewicz's matrix, cf. [4]. On the other hand, in $\underline{L}$ there are definable sentential connectives $j_1, \ldots, j_{n-2}$ such that for every valuation h in M_n

$$h(j_i(p)) = \begin{cases} 1 & \text{whenever } h(p) = i/n-1 \\ 0 & \text{otherwise,} \end{cases}$$

cf. [3].

First, it is easy to see that the conditions (N0) –(N3) hold true for C_n with $j_1, \ldots, j_{n-2}$ standing as $E_1, \ldots, E_{n-1}$, respectively. Consequently, Ł$_n$ is n-normal. In turn, the equality $\approx_{C_n} = \sim_{C_n} \cap \sim_1 \cap \ldots \cap \sim_{n-2}$ follows immediately from the fact that if for some $\alpha, \beta \in L$ $\alpha \sim_{C_n} \beta$ and $\alpha \sim_i \beta$ for each $i \in \{1, \ldots, n-1\}$, then for any valuation h in M_n $h(\alpha) = h(\beta)$. Thus, by Theorem, Ł$_n$ is n-referential.

II. Three-valued Łukasiewicz-like propositional calculus Ł$_3^I = (\underline{L}, C_3^I)$ with $I = \{1/2, 1\}$ is not m-referential for any $m \geq 2$. ($C_3^I = Cn_{M_3^I}$, where $M_3^I = (\{0, 1/2, 1\}, \rightarrow, \vee, \wedge, \neg, I)$, cf. [1]).

First, Ł$_3^I$ is not 2-referential (i.e. referential in the sense of [5]). To see this consider for example the two formulas p and $p \vee j_1(p)$ (compare I). They are C_3^I-equivalent, $C_3^I(p) = C_3^I(p \vee j_1(p))$, but at the same time their negations are not C_3^I-equivalent and therefore $\neg p \ \sim_{C_3^I} \neg(p \vee j_1(p))$ does not hold. This clearly means that the relations $\sim_{C_3^I}$ and $\approx_{C_3^I}$ do not coincide.

In turn, if in $\underline{L}$ we had sentential connective F meeting the properties (N1) and (N2) for C_3^I, then for every valuation h in $M_3^I h(F(p)) = 0$ whenever $h(p) \subseteq \{1/2, 1\}$ (by (N1)) and $h(F(F(p)) = 0$ (by (N2)). Notice that this would imply that the function of the matrix M_3^I, denote it as F^o, was constant: $F^o(x) = 0$. Consequently, contrary to (N0), $C(F(p)) = L$. Thus, Ł$_3^I$ is not 3-referential. Obviously, from that it also follows that the logic in question is not m-referential for any $m \geq 3$.

References

[1] G. Malinowski, *Degrees of maximality of Łukasiewicz-like sentential calculi*, **Studia Logica** 36:3 (1977), pp. 213–228.

[2] G. Malinowski, *Matrix referentiality*, [in:] **Frege Conference 1984**, G. Wechsung (ed. Akademie Verlag, Berlin 1984, pp. 148–161.

[3] J. B. Rosser, A. R. Turquette, **Many-Valued Logics**, North-Holland, Amsterdam, 1952.

[4] R. Wójcicki, *On matrix representations of consequence operations of Łukasiewicz's sentential calculi*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik**, 19 (1973), pp. 239–247.

[5] R. Wójcicki, *Referential matrix semantics for propositional calculi*, **Bulletin of the Section of Logic** 8/4 (1979), pp. 170–176.

[6] R. Wójcicki, **Theory of Logical Calculi. Basic Theory of Consequence Operations**, Synthese Library, 199, Kluwer Academic Publishers, Dordrecht, 1988.

Department of Logic
University of Łódź
Matejki 34a
90–237 Łódź, Poland
e-mail: gregmal@krysia.uni.lodz.pl