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NON-FREGEAN LOGIC AND OTHER FORMALIZATIONS OF PROPOSITIONAL IDENTITY

The paper is an extended version of a talk given to the XXXth Conference on the History of Logic devoted to the work of Professor Roman Suszko (Kraków, October 19-21, 1984). Its aim is to present Sentential Calculus with Identity in comparison with other formalizations of propositional identity.

1. SCI. Non-Fregean Logic is received as a result of realization of the Fregean programme [6] in pure logic without the assumption which reduces the set of denotations of sentences to two, cf. [11]. Where α is a sentence, $v(\alpha)$ its logical value i.e. either 1 (Truth) or 0 (Falsity) and $r(\alpha)$ the denotation (or referent) of α , the so-called *Fregean Axiom* may be formulated roughly as follows:

$$(FA) \quad v(\alpha) = v(\beta) \text{ implies } r(\alpha) = r(\beta).$$

Incidentally, the converse implication holds true under another assumption made by Frege.

Rejecting (FA) one breaks one-one correspondence between logical values and referents of sentences: the set of referents may in general be composed of more than two objects. The identity connective \equiv was introduced by Suszko to express coincidence and other referential relations between sentences. Obviously, \equiv is not truth-functional and its intended meaning is described by

$$(RS) \quad v(\alpha \equiv \beta) = 1 \text{ if and only if } r(\alpha) = r(\beta),$$

cf. [11].

Sentential Calculus with Identity being the most simplified version of Non-Fregean Logic is a paradigm of the classical sentential logic. SCI built

syntactically as an entailment (finitary consequence) operation is based on *Modus Ponens* rule and the axiom system comprising axioms for classical connectives $\neg, \Rightarrow, \vee, \wedge, \Leftrightarrow$, the identity axiom

$$\alpha \equiv \alpha$$

and other axioms expressing the Leibniz Law of indiscernibility of identicals:

(LZ) $v(\alpha \equiv \beta) = 1$ implies that $v[\varphi(\alpha/p) = \varphi(\beta/p)] = 1$ for any formula φ and any propositional variable p .

The last groups consists of the following axiom schemata:

$$\begin{aligned} \alpha \equiv \beta &\Rightarrow (\alpha \Leftrightarrow \beta) \\ \alpha \equiv \beta &\Rightarrow \neg\alpha \equiv \neg\beta \\ (\alpha \equiv \beta) \wedge (\gamma \equiv \delta) &\Rightarrow (\alpha \circ \gamma \equiv \beta \circ \delta) \quad \text{for } \circ \in \{\Rightarrow, \vee, \wedge, \Leftrightarrow, \equiv\}. \end{aligned}$$

Accordingly, the identity connective appearing in SCI is extensional with respect to all sentential connectives including itself.

2. Belief based identity. An extensional identity formator in a language with intensional functions was as in 1948 defined by Łoś [8]. An assertion logic of belief sentences of the form $L_x p = x$ believes that p is built in [8] to extend the classical sentential calculus in the language containing nominal variables and admitting quantification. The axiom system for belief sentences consists of the group of axioms of the form

$$L_x \alpha$$

where α is an axiom of classical sentential logic and with the following specific axioms:

- (i) $L_x(\neg p) \Leftrightarrow \neg L_x p$
- (ii) $L_x(p \Rightarrow q) \Leftrightarrow (L_x p \Rightarrow L_x q)$
- (iii) $L_x(L_x p) \Leftrightarrow L_x p$
- (iv) $\forall_x(L_x p \Rightarrow p)$.

Apart from the usual rules completing the axiomatization of sentential calculus usual rules concerning the quantifiers are accepted.

It appears that the functor $=$ defined by

$$(LI) \quad p = q =_{df} \forall_x(L_x p \Rightarrow L_x q)$$

is an extensional identity connective in the sense-mentioned in Section 1. That it is really so for classical connectives follows directly from (i)-(iv) and respective definition. The proof of self-extensionality of $=$, i.e.

$$(\alpha = \beta) \wedge (\gamma = \delta) \Rightarrow ((\alpha = \gamma) = (\beta = \delta))$$

requires additionally the use of appropriate properties of quantification.

3. Metalinguistic identity. Greniewski [7] introduced into the language of sentential logic a binary connective of *metalinguistic identity* \doteq , i.e. such a connective that “ $\alpha \doteq \beta$ ” is read as “ α is metalinguistically equivalent to β ”. The system of logic of Greniewski’s identity is built as the extension of an axiom system of classical calculus (in the language with \doteq) received by adding two following schemata:

$$\begin{aligned} \text{(A1)} \quad & ((p \doteq q) \doteq 0) \vee ((p \doteq q) \doteq 1) \\ \text{(A2)} \quad & \varphi(\alpha/p) \Rightarrow [(\alpha \doteq \beta) \Rightarrow \varphi(\beta/p)] \end{aligned}$$

and the “strengthening rule”:

$$\text{(GR)} \quad \frac{\vdash \alpha \Leftrightarrow \beta}{\vdash \alpha \doteq \beta}$$

(1 stands for any classical tautology, 0 for its negation and \vdash is usual sign asserting theses).

Due to (A2), being another formulation of Leibniz Law, \doteq is an extensional sentential connective. On the other hand, according to the prior motivation it is natural to define the modal square connective \Box in reference to \doteq . When we put

$$\Box \alpha =_{df} \alpha \doteq 1$$

then \Box is as in S5. The rule (GR) implies the *Gödel rule*

$$\text{(G)} \quad \frac{\alpha}{\Box \alpha}$$

or any of its equivalents:

$$\frac{\alpha}{\alpha \doteq 1}, \quad \frac{\alpha, \beta}{\alpha \doteq \beta}$$

Subsequently, one may show that Greniewski’s identity coincides with the strict equivalence of S5 proving that

(*) $\vdash \alpha \doteq \beta$ if and only if $\vdash \Box(\alpha \Rightarrow \beta)$ and $\vdash \Box(\beta \Rightarrow \alpha)$.

PROOF. (\Rightarrow) Assume that $\vdash \alpha \doteq \beta$. Then from $\vdash (\alpha \Leftrightarrow \alpha) \Leftrightarrow 1$ by (GR) we obtain $\vdash (\alpha \Leftrightarrow \alpha) \doteq 1$. The last formula, the assumption and (A2) imply together $\vdash (\alpha \Rightarrow \beta) \doteq 1$ and $\vdash (\beta \Rightarrow \alpha) \doteq 1$. This amounts to $\vdash \Box(\alpha \Rightarrow \beta)$, $\vdash \Box(\beta \Rightarrow \alpha)$.

(\Leftarrow) When $\vdash \Box(\alpha \Rightarrow \beta)$ and $\vdash \Box(\beta \Rightarrow \alpha)$, then $\vdash (\alpha \Rightarrow \beta) \doteq 1$, $\vdash (\beta \Rightarrow \alpha) \doteq 1$ and, therefore, $\vdash (\alpha \Rightarrow \beta) \doteq (\beta \Rightarrow \alpha)$. Notice that this yields $\vdash ((\alpha \Rightarrow \beta) \Leftrightarrow (\beta \Rightarrow \alpha))$ and, consequently, $\vdash (\alpha \Leftrightarrow \beta)$. By (GR) we finally get $\vdash \alpha \doteq \beta$.

4. Functorial calculus. The idea is due to Prior [9], [10] and was developed by Cresswell in a series of papers: [3], [4], and [5]. The functorial calculus FC is a kind of extended sentential calculus comprising variables for propositional functions f, g, h, \dots the range of which is not restricted to truth functions. In its first-order version FC is presented in [4] as a direct generalization of the classical sentential calculus in the language with implication \Rightarrow and falsum (0), general sentential quantifiers and f, g, h, \dots variables. To the classical axioms sufficient for classical sentential calculus based on \neg and 0 two axioms concerning quantification are added:

(\forall_1) $(a)A \Rightarrow B$ where A and B are wffs and B is like A except in having some wff C whenever A has the variable a provided that no variable becomes bound in obtaining B from A

(\forall_2) $A \Rightarrow B$ implies that $A \Rightarrow (a)B$, where A and B are wffs and a is a variable not free in A .

In [3] and [5] it is shown that it is possible to augment such a system as FC introducing binary connective of propositional identity as primitive with two axioms:

$$(I_1) \quad p = p$$

$$(I_2) \quad (p = q) \Rightarrow (fp \Rightarrow fq)$$

and presupposing suitable rules of uniform substitution for sentential variables. Alternatively, cf. [10], if we assume quantification over all variables, $=$ may be introduced by the following definition:

$$(DF_{=}) \quad \alpha = \beta =_{df} \quad \forall_f (f\alpha = f\beta).$$

The most obvious feature of FC is its non-extensionality. Originally, the setting up a system as PC has been motivated so as FC stands to modal systems roughly as the pure lower predicate calculus stands to first-order theories, cf. [9]. The formula

$$(EI) \quad (p \Leftrightarrow q) \Rightarrow (p = q)$$

is not a theorem of FC. The addition of (EI) would reduce FC to Leśniewskian protothetic

$$\vdash (p \Leftrightarrow q) \Rightarrow (fp \Rightarrow fq),$$

and so ultimately to the classical sentential calculus, cf. [1].

On the other hand, the rule

$$(R) \quad \frac{\alpha \Rightarrow \beta}{f\alpha \Rightarrow f\beta}$$

is not a rule of FC. Cresswell argues: “If we add the latter propositional identity would seem amount to provable equivalence. This identification suggests the possibility of defining the operator ‘It is logically necessary that p ’ (Lp) as ‘ p is identical with some provable truth’ where we choose some theorem (call it 1) and define $L\alpha =_{df} (\alpha = 1)$ ”, cf. [3], p. 191.

It is proved in [3] that the extension FCR of FC is deductively equivalent to the modal system S4. Later on, in [5], Cresswell showed that S5 can also be obtained as an axiomatic strengthening of FCR.

5. Final remarks. It is evident that any comparative question concerning the logic of propositional identity may be posed either in reference to a particular language or to a special feature of a formalization.

Among several current requirements to three following seem to be of a particular importance:

(1) Extensionality in the sense of Leibniz Law of indiscernibility of identicals

(2) Formal character of identity nothing except general properties such as e.g. reflexivity, symmetry or transitivity has either be assumed or proved

(3) Purely sentential character of formalization: the language has to contain only sentential variables.

We notice that all systems of identity considered here except Cresswell’s FC satisfy (1). For systems like FCR, FCS5 and for Greniewski’s

system this fact may be interpreted as that “the strict equivalence connective behaves in S_4 and stronger modal systems as extensional identity connective”, cf. [11] and Section 3. Condition (2) is satisfied for FC, SCI and (in a sense) for Łoś system. Finally, (3) holds true in Greniewski’s system and SCI.

If one agreed that all the properties (1)-(3) are basic for the logic of propositional identity, SCI would be considered as the only genuine logic of this kind, [The property that logics of identity corresponding to S_4 and S_5 proved to be axiomatic strengthening of SCI, cf. [1] and [12], supports the conclusion. Actually, it turned out that some systems of non-pure extensional identity may be received without the use of intensional rules such as (GR) or (R).]

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