**B**ulletin of the **S**ection of **L**ogic
Volume 12/3 (1983), pp. 90–96
reedition 2008 [original edition, pp. 90–98]

Grzegorz Malinowski

# PSEUDO-REFERENTIAL MATRIX SEMANTICS FOR PROPOSITIONAL LOGICS

Dedicated to
Professor Ryszard Wójcicki
on his successive birthday

Referential matrix semantics of R. Wójcicki [5] and [4] is extended to cover the class of all structural propositional calculi.

## 0. Clause

$\underline{L}$ is used to denote a propositional language (an algebra of formulas with $L$ being the set of its elements. $C$ is a structural consequence operation and just in that sense $(\underline{L}, C)$ is called a *propositional logic*. All other notation is either borrowed from [3] or it already appears in other works we are referring to.

## 1. Pseudo-referential matrix

Given a non-empty set $T$ if indices (points of reference, possible worlds etc., cf. [5]) and a subset $T^* \subseteq 2^T$ a *pseudo-referential matrix* for $\underline{L}$ relative to $(T, T^*)$ is a pair

$$W = (\underline{A}, D)$$

where

$(w_1)$ $\underline{A}$ is an abstract algebra similar to $\underline{L}$ such that $A$ is a subset of $\{0,1\}^T$ i.e. elements of $A$ are mapping from $T$ into $\{0,1\}$

$(w_2)$ $D = \{\{r \in A : \exists_{t \in t^*} r(t) = 1\} : t^* \in T^*\}$.

Notice that the generalization of referentiality along these lines merits in choice of the set of distinguished objects, cf. [5], [6]. The following fact concerning consequence operations $Cn_W$ resembles similar result for referential matrices and can be checked by an easy computation:

PROPOSITION. *For a pseudo-referential matrix $W$ for $\underline{L}$ relative to $(T, T^*)$ the following are equivalent:*

*(i)* $\alpha \in Cn_W(X)$

*(ii) For each* $h \in Hom(\underline{L}, \underline{A})$ *and each* $t^* \in T^*$, $\exists_{t \in t^*}(h\alpha)(t) = 1$ *whenever* $\exists_{t \in t^*}(h\beta)(t) = 1$ *for all* $\beta \in X$.

Now, we are ready to prove

THEOREM. *For every propositional logic $(\underline{L}, C)$ there exists a pseudo-referential matrix $W_C$ such that*

*(∗)* $C = Cn_{W_C}$.

PROOF. (An outline.) Consider the Lindenbaum's bundle $\alpha_C = (\underline{L}, \{C(X) : X \subseteq L\})$, or, equivalently, the class of matrices $L_C = \{(\underline{L}, C(X)) : X \subseteq L\}$. Then cf. [3],

(•) $C = Cn_{\alpha_C} = Cn_{L_C}$.

For each $\alpha \in L$ let $f_\alpha$ denote the function from $L$ to $\{0,1\}$, $f_\alpha \in \{0,1\}^L$, characteristic of $\{\alpha\}$ i.e. defined as follows:

$$f_\alpha(\gamma) = \begin{cases} 1 & \text{whenever} \quad \gamma = \alpha \\ 0 & \textit{otherwise.} \end{cases}$$

The set of all such $f_\alpha s', A_L = \{f_\alpha : \alpha \in L\}$ is closed under the operations corresponding to propositional connectives of $\underline{L}$ in the natural way:

$$F(f_{\alpha_1}, \dots, f_{\alpha_n}) = f_F(\alpha_1, \dots, \alpha_n)$$

for a $n$-ary $(n \geq 1)$ connective $F$. Consequently, the algebra $\underline{A}_L$ having $A_L$ as the set of elements and all associates of propositional connectives of $\underline{L}$ as operations is similar to $\underline{L}$. Obviously, $\underline{A}_L$ is a referential algebra with $T = L$.

Now, it is important to notice the following ($e$)-property of $\underline{A}_L$: with each homomorphism sending $L$ into $A$, $h \in Hom(\underline{L}, \underline{A}_L)$, a substitution $e \in End(\underline{L})$ is associated such that $h\alpha = f_{e\alpha}$ for any formula $\alpha$ (proof by an induction).

Putting $T^* = \{C(X) : X \subseteq L\}$ we finally get a pseudo-referential matrix $W_C = (\underline{A}_L, D_C)$ with $D_C = \{\{f_\alpha : \alpha \in C(X)\} : X \subseteq L\}$. That $W_C$ is strongly adequate for $C$ i.e. that ($*$) holds true may be verified by the use of Proposition, ($\bullet$) and the ($e$)-property of $\underline{A}_L$.

## 2. Self-extensionality

If $T^* = \{\{t\} : t \in T\}$ a pseudo-referential matrix to $(T, T^*)$ becomes *referential*, cf. [5], [4]. Recall that for a logic $(\underline{L}, C)$ a referential matrix adequate exists if and only if $C$ is *self-extensional* i.e. if

($s$) $C(\alpha) = C(\beta)$ implies $C(\varphi[\alpha/p]) = C(\varphi[\beta/p])$

for any formulas $\alpha, \beta, \varphi$ of $\underline{L}$ and any propositional variable $p$.

Now, we are going to produce the "canonical" reference matrix for a self-extensional $C$ (! already constructed in [6]) form the pseudo-reference matrix appearing in the proof of Theorem. Our aim is to show how congruences of (pseudo-) referential matrices do their job.

Assume then that with a self-extensional $C$ we are given the matrix $W_C$ constructed in the proof of Theorem. On $A_L$ we define the relation $\equiv_C$ as follows:

$$f_\alpha \equiv_C f_\beta \text{ iff } \forall_{X \subseteq L}(\alpha \in C(X) \text{ iff } \beta \in C(X)).$$

$\equiv_C$ is an equivalence relation and therefore we put

$$f_\alpha = ||f_\alpha|| = \{f_\beta : f_\beta \equiv_C f_\alpha\}.$$

In order, it is crucial to observe that each $f_\alpha$ can be viewed as mapping of $\{C(X) : X \subseteq L\}$ into $\{0, 1\}$, so that

$f_\alpha \in \{0, 1\}^{\{C(X) : X \subseteq L\}}$

and that

$$f_\alpha(C(X)) = \begin{cases} 1 & \text{if } f_\beta \in f_\alpha \quad \text{for some } \beta \in C(X) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the assumption that $C$ is self-extensional yields that $\equiv_C$ is a congruence on $A$ i.e. that

$(s')$ $f_\alpha \equiv_C f_\beta$ implies $f_{\varphi[\alpha/p]} \equiv_C f_{\varphi[\beta/p]}$.

A moment's reflection shows that

$$D_{L/\equiv_C} = \{\{f_\alpha \in A_{L/\equiv_C} : f_\alpha(C(X)) = 1\} : X \subseteq L\}$$

what in turn means that the quotient

$$W_{L/\equiv_C} = (\underline{A}_{L/\equiv_C}, D_{L/\equiv_C})$$

is a *reference* matrix for $\underline{L}$.

To complete the deal it suffices to show that $\equiv_C$ is a matrix congruence relation for $(\underline{A}, D)$. For if for some $\alpha_1, \alpha_2 \in L$ such that $f_{\alpha_1} \in D_{t^*}$ and $f_{\alpha_2} \in D_{t^*}$ for some $D_{t^*} = \{f_\alpha : \alpha \in C(X)\}$ (given $X \subseteq L$) we had

$f\alpha_1 \equiv_C f\alpha_2$

then for any $Y \subseteq L$ we would also have

$\alpha_1 \in C(Y)$ iff $\alpha_2 \in C(Y)$

what actually is not the case for $Y = X$.

Thus, $W_{L/\equiv_C}$ is equivalent to $W_L$, cf. [3]. The former is the referential matrix for $C$ we were looking for – compare [6].

## 3. Algebraic semantics

$C$ with $C(\emptyset) \neq \emptyset$ has an algebraic semantics, $C = Cn_K$ for a class of matrices $K = \{(A_i, \{a_1\}) : i \in I\}$ with single elements designated, if and only if for any $X \subseteq L$, $\alpha \in L$ such that $\alpha \in C(X)$

$(a)$ $C(X) = \{\beta : \beta \approx^C_X \alpha\}$,

where $\approx^C_X$ is defined (relatively to $X$ and $C$) as follows:

$$\alpha_1 \approx^C_X \alpha_2 \text{ iff } C(X, \varphi[\alpha/p]) = C(X, \varphi[\beta/p]),$$

any $\alpha_1, \alpha_2, \varphi \in L$ and any variable $p$, cf. [2], [1].

In what follows we are dealing with a logic $\underline{L}, C)$, $C$ with $C(\emptyset) \neq \emptyset$ and having algebraic semantics. As in the preceding Section we produce appropriately reduced "canonical" pseudo-reference matrix.

Our first move is to split a given pseudo-reference semantics. Each $(T, T^*)$ pseudo-referential matrix $W = (\underline{A}, D)$ with $D = \{D_{t^*} : t^* \in T^*\}$ can be replaced by the following boundle of matrices:

$$\mathbf{W} = \{(\underline{A}, D_{t^*}) : t^* \in T^*\}.$$

($Cn_W = Cn_{\mathbf{W}}$, cf. [3]). That in the case of the matrix appearing in the proof of Theorem leads to

$$\mathbf{W}_C = \{(\underline{A}_L, \{f_\alpha : \alpha \in C(X)\}) : X \subseteq L\}.$$

Subsequently, we take the relations $\equiv^C_X$ defined on $A_L$ (relatively to $X \subseteq L$) by

$$f_\alpha \equiv^C_X f_\beta \text{ iff } \alpha \approx^C_X \beta.$$

As each $\approx^C_X$ is a congruence on $\underline{A}_L$. The former is matrix congruence of the matrix $(\underline{L}, C(X))$ and, consequently, the latter is a congruence of $(\underline{A}_L, \{f_\alpha : \alpha \in C(X)\})$. So, each matrix $(\underline{A}_L, \{f_\alpha : \alpha \in C(X)\})$ is equivalent to the quotient

$$(\underline{A}_{L/\equiv^C_X} : \{f_\alpha \in C(X)\}/\equiv^C_X)$$

and therefore $W_C$ can be replaced by

$$\mathbf{W}_{C/\approx} = \{(\underline{A}_{L/\equiv^C_X}, \{f_\alpha : \alpha \in C(X)\}/\equiv^C_X) : X \subseteq L\}.$$

Notice that, under our assumption, by (a), $\{f_\alpha : \alpha \in C(X)\}/\equiv^C_X$ is always a *one element* set! The final step consists in "pasting together", cf. [6], all (pseudo-) referential matrices from $\mathbf{W}_{C/\approx}$ to get a single adequate matrix for $C$, which in the case considered is as much special as it has only one-element sets of indices designated: $T^* \subseteq \{\{t\} : t \in T\}$ – so it is in a sense *discrete*.

## 4. Postscript

What has been presented here may, more or less precisely, be conceived as a heap of variations on the reference matrix semantics theme composed by R. Wójcicki [5] and further elaborated by himself in [4] and [6]. On the other hand, the main result indicated as Theorem evidently is a referential counterpart of another important Wójcicki's result, namely the completeness property of propositional logics (•) mentioned at the very beginning of our proof.

Two further remarks seem to be in order. The first of them concerns technology – it is our conviction that such reduction procedures as those used in Section 2 and Section 3, looking perhaps at first glance either painfully or trivially, if appropriately interpreted may bring a slightly deeper insight into important semantical properties of propositional logics. The "referential point of view" shows, for example, that two kinds of logics considered above are in some way related and, at the same time, indicates some differences. In both cases, the "canonical" matrices are *discrete* in the sense mentioned in Section 3. The difference is that for self-extensional logic $T^* = \{\{t\} : \ t \in T\}$ while for a logic having algebraic semantics $T^*$ can be properly included in $\{\{t\} : \ t \in T\}$.

Secondly, the set $D$ of distinguished elements of a pseudo-referential matrix looks perhaps also somewhat strange. If, however, one realizes that actually we are dealing with a *two-valued* semantics *in disguise*, then the assumption concerning $T^*$ ($T^* \subseteq 2^T$ and *not* $T^* \subseteq \{\{t\} : \ t \in T\}$ in general) may be understood as forcing a *super-valuation* which notion is widely accepted by scholars in logic. Incidentally, would not *super-referential* be a more appropriate adjective for the kind of matrices considered?

Obviously, the idea of pseudo-referential semantics (referential as well) can be interpreted as a Kripke-style semantics. That such an approach opens (too large, I am afraid!) variety of possible philosophical interpretations is easily visible.

## References

[1] J. Czelakowski, *Equivalential logics (II)*, **Studia Logica** 40 (1981), no. 4, pp. 355–372.

[2] G. Malinowski, *A proof of Ryszard Wójciki's conjecture*, this **Bulletin** 7, no. 1 (1978), pp. 20–25.

[3] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica** 33 (1973), pp. 7–37.

[4] R. Wójcicki, *More about referential matrices*, this **Bulletin** 9, no. 2 (1980), pp. 93–95.

[5] R. Wójcicki, *Referential matrix semantics for propositional calculi*, this **Bulletin** 8, no. 4 (1979), pp. 170–176.

[6] R. Wójcicki, **Theory of propositional calculi** (an introduction), Ossolineum, in print.

*Łódź University*
*90-950 Łódź*
*Poland*