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EXTENDING ATOMISTIC FRAMES
PART II

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Abstract
The paper concludes an earlier one (*Logica Trianguli*, 5) on extensions of atomistic semantic frames. Three kinds of extension are considered: the adjunctive, the conjunctive, and the disjunctive one. Some theorems are proved on extending "Humean" frames, i.e. such that the elementary situations constituting their universa are separated by the maximally coherent sets of them ("realizations").

1. Preamble

This is the conclusion of [1] and [2]. There, and also in [3], we have introduced the concept of a "semantic frame", as based on the notions of an *elementary situation*, and of such situations being the *verifiers* of propositions. The *ontology* of such a frame is the couple (SE, $\boldsymbol{R}$) where SE is the set – otherwise arbitrary – of all elementary situations, and $\boldsymbol{R}$ is a collection the members of which are *realizations* (or "possible worlds") as characterized by the following five axioms:

(R1) $\boldsymbol{R} \subset P(SE)$

(R2) $\boldsymbol{R} \neq \varnothing$

(R3) $\cup\boldsymbol{R} \neq SE$

(R4) $\cap\boldsymbol{R} \neq \varnothing$

(R5) $R_1 \subset R_2 \Rightarrow R_1 = R_2$, for all $R_1, R_1 \in \boldsymbol{R}$

In language realizations are represented by complete theories, i.e. by sets of propositions maximal under classic consequence.

Setting now r(x) = {R ∈ $\boldsymbol{R}$: x ∈ R}, for all $x \in SE$, we see that the collection $\boldsymbol{R}$ induces in SE a quasi-ordering of *entailment*:

$x \vdash y$ iff $r(x) \subset r(y)$;

and that under the equivalence

$$x \sim_r y \text{ iff } r(x) = r(y)$$

the partition *SE/r* is then a bounded partial ordering, with

$$/x/ \leq /y/ \text{ iff } y \vdash x .$$

The set of all *impossible* situations Λ(lambda) = *SE* - $\cup\boldsymbol{R}$ is the unit of that ordering, and the set of all the *necessary* ones Ω (omega) = $\cap\boldsymbol{R}$ s its zero. The rest, if any, are the *contingent* ones.

To the basic ontological axioms R1 – R5, constitutive to any semantic frame, we add successively further ones, obtaining thus certain special kinds of frames. Firstly the following two:

(R6) $\wedge A \in \text{Fin}\, SE \ \vee z \in SE \ \wedge R \in \boldsymbol{R}\!: z \in R \text{ iff } A \subset R.$

(R7) $(\wedge B \in \text{Fin}\, A \ \vee R \in \boldsymbol{R}\!: B \subset R) \Rightarrow \ \vee R' \in \boldsymbol{R}\!: A \subset R'$;
for all $A \subset SE$.

Under R6 the frame becomes *conjunctive*, which means that its partial ordering *SE/r* is turned into a j*oin-semilattice* then. ("Fin *A*" denotes all the finite subsets of *A*, including the empty one.) And R7 makes the frame ***R***-*compact*; members of the collection $\{R/r\!: R \in \boldsymbol{R}\}$ are then all the *maximal ideals* of the join-semilattice *SE/r.*

Next we adopt two axioms of *atomicity*. An elementary situation *x* is a *quasi-atom* (written: $x \in QA$) iff it is not necessary and any non-necessary situation entailed by it is *r*-equivalent to it. Let "$At(x)$" denote all the quasi-atoms entailed by the elementary situation *x*. We assume then: for any $x \in SE$,

(R8) $x \notin \Omega \Rightarrow At(x) \neq \varnothing$, and $x \notin \Lambda \Rightarrow At(x) \in \text{Fin}\, QA$

(R9) $\wedge R \in \boldsymbol{R}\!: At(x) \subset R \Rightarrow x \in \mathrm{R}.$

In view of R8 we say that the quasi-ordering is *finitely atomic*, and in view of R9 that it is *atomically determinate*. In view of both the ordering *SE/r* is a *finitely atomistic lattice*. (I.e., every non-unit member of it is the join of a finite number of its atoms.)

Matters are greatly simplified if we adopt also the following axiom which obliterates the difference between the orderings of *SE* and of *SE/r:*

(H) $\wedge x,y \in SE\!: r(x) = r(y) \Rightarrow x = y.$

Frames satisfying axiom H we shall call *Humean*, for it is put forward in Hume's "Treatise" as the logical mainstay of the metaphysical system presented there. By that axiom the members of *SE* are *separated* by those of ***R***, which makes the ordering of *SE isomorphic* to that of *SE/r*. (As each *r*-equivalence class is a singleton then.) So the quasi-ordering *SE* becomes then a finitely atomistic lattice itself, with *QA* being its atoms, and with $\Omega = \{o\}$, $\Lambda = \{\lambda\}$, for some $o, \lambda \in SE$. Thus in a Humean semantic frame there is just one necessary situation, and just one impossible.

To describe extensions of atomistic frames properly, we have to confine our set-up yet more. Presumably it could be done in a more general set-up too, but that has proved too hard to us. Thus we assume, moreover, that on top of being Humean the frames satisfy also the following two axioms:

(R10) The frames are dimensionally determinate.
(R11) The frames are *uniquely* atomistic.

The former is to say that the atoms *QA* of the finitely atomistic lattice *SE* are partitioned into blocks $D_i \in \boldsymbol{D}$ – to be called the *logical dimensions* of the frame – such that distinct atoms in one block always exclude each other, and every block is intersected by all the members of ***R***. (Clearly each intersection is a singleton then.)

And being "uniquely atomistic" means this: if an elementary situation is possible, i.e. $x \neq \lambda$, and it is the join of a set of atoms A, i.e. $x = \sup A$, then $A = At(x)$. No lesser set will do. Thus no possible situation must be the join of different sets of atoms, as it is e.g. the case in the modular sublattice shown in Figure 1, where $x = x_1 \vee x_2 \vee x_3 = x_1 \vee x_2 = \ldots$.

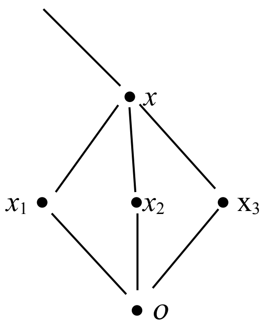

Figure 1

As has been shown in [3], p. 310, the following holds for all lattices:

(1) Suppose a lattice is finitely atomistic. Then it is uniquely atomistic

if and only if it is conditionally distributive.

And by being *conditionally distributive* we mean that in that lattice the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds only under the proviso the $y \vee z \neq 1$.

2. Adjunctive Extensions

Let F, F' be two dimensionally determinate Humean frames, both finitely and uniquely atomistic; with QA, QA' as their respective sets of atoms, and $\boldsymbol{D}$, $\boldsymbol{D}'$ as their logical dimensions. In [2] we have called F' an *adjunctive extension* of F if $\boldsymbol{D} = \boldsymbol{D}' - \{D'\}$, for some $D' \in \boldsymbol{D}'$; i.e., if $QA' = QA \cup D'$, with $QA \cap D' = \varnothing$. Thus F' differes from F only by a new dimension having been adjoined to it. We have shown there the following to hold then for the respective lattices of elementary situations SE, SE':

(1) If F' is an adjunctive extension of F, then SE is a $\{0,1\}$-sublattice of SE'.

Moreover, let x,y be two elementary situations. We call them *compossible* if $x \vee y \neq \lambda$, and overlapping if $x \wedge y \neq o$. Now let A, B be two non-empty sets of elementary situations. We call them orthogonal to each other if cross-wise all their members are compossible, but never overlap; i.e., if $x \vee y \neq \lambda$, and $x \wedge y \neq o$, for all $x \in A$, $y \in B$.

Observe that if A is orthogonal to B, then λ cannot be member of either. (As otherwise the universal cross-wise compossiblity of their members would be violated.) And they must be disjoint up to the zero-element o, i.e.: for any $A,B \subset SE$,

(2) If A is orthogonal to B, then either $A \cap B = \varnothing$, or $A \cap B = \{o\}$.

Indeed, suppose $A \cap B \neq \varnothing$. So $z \in A \cap B$, for some $z \in SE$. But if $z \neq o$, then $z \wedge z \neq o$, which contradicts orthogonality, as both $z \in A$ and $z \in B$.

Using the operation $A \bullet B = \{x \vee y \in SE\colon\ x \in A, y \in B\}$, usually called the *product of subsets*, we have then as shown in [2]:

(3) If in the adjunctive extension of F to F' the extending dimension $D' \in \boldsymbol{D}'$ is orthogonal to the lattice SE of F (i.e., to the set $SE - \{\lambda\}$), then the maximal ideals $R' \in \boldsymbol{R}'$ of the lattice SE' in F' are all of the form $R' = R \cup R \bullet \{t\}$, for some $t \in D'$, $R \in \boldsymbol{R}$.

3. **Conjunctive Extensions**

To begin with, we shall construct an example displaying the idea in question. Suppose the object of our observation to be the square X of Figure 2, changing often and abruptly in colour, but to our perception always coloured uniformly all over. Suppose it to be so minute as to lie at the border of our powers of discrimination: we perceive it only as a single speck. And suppose finally that at different times we have seen it taking one of the following four neutral hues only: black, dark grey, light grey, or white; and that somehow we know it can take no others.

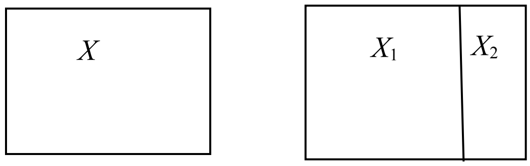

Figure 2

By the nature of colour those four hues are mutually *exclusive*: at any given moment one at most can obtain. And by supposition they are between them *exhaustive*: at the moment given one at least must obtain. Thus we have here four atomic situations at hand – i.e. four possible states of the square X – which may be represented by couples of the form (square, hue). These atomic situations constitute a logical dimension:

$$D = \{(X, B), (X, dG), (X, lG), (X, W)\}$$

of some unspecified semantic frame F.

Now suppose that by some progress in the methods of observation, say by the invention of a new instrument, our powers of discrimination have been increased. We can see the square X actually to consist of two unequal rectangles X_1 and X_2, as again in Figure 2. And we observe that each of them takes just two hues: either black, or white. So it turns out that square X appears to our senses as black if under the new instrument both rectangles show black; as dark grey if the bigger one shows black and the smaller one white; as light grey if the opposite is the case; and as white if both show white.

Thus in the wider frame F' corresponding to our improved means of observation the would-be atomic situations of the frame F turn out to be compounds of simpler ones, which are the atoms of F'. For we have then:

$$(X, B) \quad = (X_1,B) \vee (X_2,B)$$
$$(X, dG) \quad = (X_1,B) \vee (X_2,W)$$

$(X, lG) \quad = (X_1,W) \vee (X_2,B)$
$(X, W) \quad = (X_1,W) \vee (X_2,W)$

with the sign "$\vee$" being here, of course, not that of a propositional connective, but that of the algebraic join-operation in *F'*: atoms of the old frame *F* cease to be atoms in the new one *F'*. Observe that they are partitioned into logical dimensions all right:

$D_1' = \{(X_1,B)\ , (X_1,W)\}$
$D_2' = \{(X_2,B)\ , (X_2,W)\}$,

as evidently both are exclusive and exhaustive. Thus, eventually we get: $D = D_1' \bullet D_2'$, as the two latter ones are orthogonal to each other by construction. (Note that no other partition is feasible here, for $\{\{(X_1,B)\ , (X_2,B)\}, \{(X_1,W)\ , (X_2,B)\}\}$ does not yield any logical dimensions as in each block its members are not mutually exclusive. This, however, is not generally so: a partition into logical dimensions need not be unique.)

Our humble example leads to a general definition. Let $\boldsymbol{D}'$ be the dimensions of a given frame *F'*, and $\boldsymbol{D}_i'$ – a finite non-empty subcollection of them: $\boldsymbol{D}_i' \subset \boldsymbol{D}'$. Moreover, let all the dimensions in $\boldsymbol{D}_i'$ be *orthogonal throughout*, i.e. not just in pairs, but so that with $\boldsymbol{D}_i' = \{D_1'\ , D_2'\ , ..., D_k'\}$ we have $t_1 \vee t_2 \vee ... \vee t_k \neq \lambda$ for every selection of atoms $\{t_1, t_2, ... , t_k\} \subset QA'$, taken each from a different dimension. We define then:

(4) The frame *F'* is a *conjunctive extension* of the frame *F* if for some dimension *D* of *F* there is a finite number of dimensions $D_1'\ , D_2'\ , ..., D_k'$ in *F'* such that $QA' = (QA - D) \cup (D_1' \cup D_2' \cup ... \cup D_k')$ and $D = D_1' \bullet D_2' \bullet ... \bullet D_k'$.

Now the question arises: how, under a conjunctive extension of a frame *F* to the frame *F'*, are their respective ontologies (SE, $\boldsymbol{R}$) and (SE, $\boldsymbol{R}'$) related to one another? To answer it we shall for the sake of simplicity consider merely the case k = 2, i.e. that of $\boldsymbol{D}_i' = \{D_1', D_2'\}$. It is then relatively easy to generalize it to an arbitrary number of dimensions $k \geq 2$.

We start with forming two subsets of the lattice SE', to be denoted by "X^+" and "X^-", and determined by the dimensions D_1' and D_2', i.e. by $\boldsymbol{D}_i'$. One consists of all the members of SE' which contain atoms of both D_1' and D_2'; the other of all those which do not contain atoms of either. I.e., we set:

$X^+ = \{x \in SE' : \vee\, t_1 \in D_1' : t_1 \leq x \text{ and } \vee\, t_2 \in D_2' : t_2 \leq x\}$;
$X^- = \{x \in SE' : \neg \vee\, t_1 \in D_1' : t_1 \leq x \text{ and } \neg \vee\, t_2 \in D_2' : t_2 \leq x\}$

Observe that the unit λ belongs to X^+, though not to X^-; and conversely for the zero o.

Then again we have the same relationship:

(5) If the frame F' is a conjunctive extension of a frame F, then the lattice SE of F is a $\{0,1\}$-sublattice in the lattice SE' of F'; with $SE = X^+ \cup X^-$.

Indeed, take any $x, y \in SE$ and consider first their join $x \vee y = z$. If $z = \lambda$, then $z \in X^+$; so suppose $z \neq \lambda$. Now if both x and y are in X^+, then obviously so is their join z. Similarly if x is in X^+ and y is in X^-, or conversely. Suppose therefore that both are in X^-, i.e. neither contains any atom of either D_1' or D_2': $(At(x) \cup At(y)) \cap (D_1' \cup D_2') = \varnothing$. Recall that by the axioms R8, R9, and R11 the lattice SE' in question if finitely and uniquely atomistic; hence the following implication holds for it, as shown in [3], p. 309: if $x \vee y \neq \lambda$, then $At(x \vee y) = At(x) \cup At(y)$. But by supposition the union $At(x) \cup At(y)$ contains no atom of either D_1' or D_2', so in view of that implication neither does the set $At(x \vee y)$. Thus $z \in X^-$, and we see eventually that the set SE is closed under the joins of the lattice SE'.

Closure under the meet of SE' is obvious. If $x \wedge y = o$, then $x \wedge y \in X^-$. If either x, or y is in X^-, their meet must be there too. And if both are in X^+, so is their meet. QED.

4. Some Lemmas and Definitions

The relationship between the collections ***R*** and ***R'*** is a more intricate question. Before addressing it directly we introduce several lemmas and definitions with regard to the sort of semilattice relevant here, especially to their ideals. Firstly we have this:

(6) Let L be a $\{o\}$-sublattice of the bounded lattice L'; and let I' be an ideal of the latter. Then the set $I = I' \cap L$ is an ideal of L.

Indeed, I is never empty, as the zero o always belongs both to I' and L. Obviously, I is a $\{o\}$-sublattice of L' too; for I', being an ideal of L', is also a $\{o\}$-sublattice of it, and the intersection of two sublattices is a sublattice again. So take any $y \in I$, and consider an $x \in L$ such that $x \leq y$. Then $x \in I'$, as y is also in I', and the latter is an ideal. Thus $x \in I' \cap L$ all right.

Secondly we have:

(7) Let I be an ideal of L, as above in (6); and let $I'(I)$ be the ideal of L' generated by the set I viewed as a subset of L'. Then $I = I'(I) \cap L$.

Indeed, inclusion $\subset$ is obvious. Conversely take any $z \in L'$ such that $z \in I'(I)$ and $z \in L$. By the former – cf. [4], p. 18, "Lemma 1(ii)" – we have: $z \leq x_1 \vee ... \vee x_k$, for some $x_1, ..., x_k \in I$. Hence, as z is in L, it must be in I too. And thirdly:

(8) Let L be an arbitrary join-semilattice with unit, and let M be a proper ideal of L. Then the ideal M is maximal if and only if, for any $x \in L$: $x \notin M \Rightarrow \bigvee z \in M$: $x \vee z = 1$.

Thus an ideal is maximal then if and only if to any element outside of it there is an *impedance* inside it, i.e. another element incompatible with the former.

Indeed, the "only-if" part of (8) is "Proposition 15" of [3], p. 311. (Or the parallel "Proposition 3" there, p. 185, for the quasi-ordering SE of an arbitrary semantic frame satisfying axioms R1 – R7.) As for the "if" part, suppose M is not maximal. There is then a proper ideal I comprising it: $M \subset I$, though $M \neq I$. Thus we have an $y \in L$ such that $y \in I$ and $y \notin M$. By assumption, as y is outside of M, there must be then an $z \in M$ such that $y \vee z = 1$. But as I is an ideal, this implies $1 \in I$, which contradicts its being proper.

In Section 1 we have called a semantic frame "Humean" if its set SE of elementary situations is *separated* by the collection of realizations $\boldsymbol{R}$; i.e., if for any $x,y \in SE$ we have:

$$x \neq y \Rightarrow \bigvee R \in \boldsymbol{R}: (x \in R \text{ and } y \notin R) \text{ or } (x \notin R \text{ and } y \in R).$$

We have also defined on SE a quasi-ordering of *entailment* "$\vdash$": for any $x,y \in SE$,

$$x \vdash y \quad \text{iff} \quad \bigwedge R \in \boldsymbol{R}: x \in R \Rightarrow y \in R.$$

The same definitions are applicable to arbitrary join-semilattices with unit – and to such lattices too, of course – if $\boldsymbol{R}$ is interpreted as the collection of their maximal ideals.

So let L be an arbitrary join-semilattice with unit, and let $\boldsymbol{M}$ be the collection of all its maximal ideals. Then the following holds:

(9) L is separated iff $\bigwedge x,y \in L$: $x \vdash y \Rightarrow x \leq y$.

For proof write the implication in (9) without abbreviations:

$$\bigwedge M \in \boldsymbol{M}\, (y \in M \Rightarrow x \in M) \Rightarrow x \leq y.$$

And consider it in its transposed form:

$$\neg (x \leq y) \Rightarrow \vee M \in \boldsymbol{M}: y \in M \text{ and } x \notin M.$$

This is sufficient for L to be separated, as assuming that $x \neq y$ we get the disjunction: $\neg\, x \leq y$ or $\neg\, y \leq x$. The first disjunct is then the antecedent of the transposed implication, and the second one is that of the same under substitutions x/y and y/x there. Thus in disjunction they yield the consequent of the implication by which separation of L by $\boldsymbol{M}$ is defined.

It is also necessary, as $\neg\, x \leq y$ means $x \vee y \neq y$. By assumption, L is separated. Hence, for some $M \in \boldsymbol{M}$, we have either $y \in M$ and $x \vee y \notin M$, or conversely. The latter, however, is a patent impossibility, so we are left with former. But if $x \vee y \notin M$, then either $x \notin M$ or $y \notin M$, as M is an ideal. The latter again is out of the question as flatly contradicted by the conclusion just reached. Thus eventually: $y \in M$ and $x \notin M$, for some $M \in \boldsymbol{M}$. QED.

So far we have considered entailments only of the "element-to-element" type, but that may be generalized. (Cf. [3], Section 3.3 "Entailments and Independence in Join-Semillatices", pp. 134-136.) The simplest generalization is to define a "set-to-element" type of entailment as follows: for any $A \subset L$, $y \in L$:

$$A \vdash y \quad \text{iff} \quad \wedge M \in \boldsymbol{M}: A \subset M \Rightarrow x \in M.$$

The "element-to-element" type is just a special case, with "$x \vdash y$" taken as abbreviation of the formula "$\{x\} \vdash y$".

Let us introduce one more notion. The "set-to-element" type of entailment in L is said to be *finite* if for any $A \subset L$, $y \in L$: $A \vdash y$ implies that $B \vdash y$, for some $B \in Fin\ A$.

Now we are ready for a theorem:

(10) Let L be a separated join-semilattice with unit, and let its "set-to-element" entailment be finite. Then if I is an ideal of L, we have: $I = \bigcap \{M \in \boldsymbol{M}: I \subset M\}$.

Indeed, inclusion $\subset$ is obvious. So suppose the opposite one does not hold: there is an element $y \in \bigcap$ such that $y \notin I$. By assumption, however, that element appears in every maximal ideal M conatining the set I, as its "satelite" so to say. But this means that it is entailed by that set: $I \vdash y$. But entailment is finite here, so there must be a set $A \in Fin\ I$ such that $A \vdash y$. But L is separated, so by (9) the latter means that $y \leq x$. Consequently $y \in I$, which is a contradiction.

Finally observe yet the following lemma:

(11) Let L be a $\{1\}$-subsemilattice of the join-semilattice L', and let M be a maximal ideal of L. Then for any maximal ideal M' of L such that $M \subset M'$ we have: $M' \cap L = M$.

Indeed, suppose this be not so. Then we should get a set-up like that of Figure 3, with an $y \in M'$ such that $y \in L$ but $y \notin M$. So in view of lemma (8) there would be an $x \in M$ such that $x \vee y = 1$. For the set M is a maximal ideal of L, and y though being in L, is not in M. But then we should have: $x,y \in M'$, i.e. $1 \in M'$, as M' is an ideal – which is a contradiction.

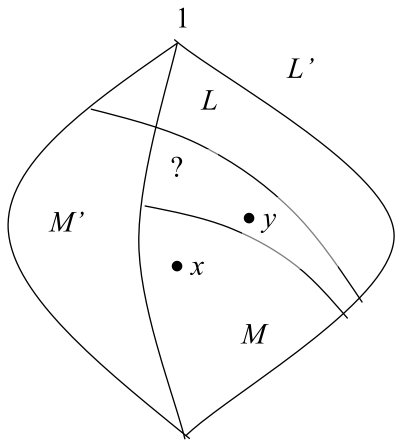

Figure 3

5. Realizations under Conjunctive Extension

In order to express succinctly the relationship between $\boldsymbol{R}$ and $\boldsymbol{R}'$ when the latter is a conjunctive extension of the former, we define an operation on collections of sets. Let U be an arbitrary subset of some fixed set U', and let $\boldsymbol{X}'$ be a non-empty collection of subsets of the latter. Then we set:

$$\boldsymbol{X}'/U = \{X' \cap U: X' \in \boldsymbol{X}'\} - \{\varnothing\}.$$

Thus $\boldsymbol{X}'/U$ is the *restriction* of a collection $\boldsymbol{X}' \subset P(U')$ to the collection $\boldsymbol{X} \subset P(U)$ of the non-empty restrictions of members of the former to the set $U \subset U'$.

Moreover, let henceforth $D(x)$ be the collection of all the dimensions to which any atoms of the element x belong. I.e.,

$$D(x) = \{ D \in \boldsymbol{D}: At(x) \cap D \neq \varnothing\},$$

with $At(x) = \{t \in QA: t \leq x\}$.

And consider yet another lemma, to the effect that under throughout orthogonality elements are *incompatible by atoms only*:

(12) Let L be a bounded lattice, finitely atomistic and dimensionally determinate, with dimensions orthogonal throughout. Take any $x,y \neq 1$. If $x \vee y = 1$, then for some atoms t_1, t_2 we have: $t_1 \leq x$, $t_2 \leq y$, and $t_1 \vee t_2 = 1$.

Clearly $D(t_1) = D(t_2)$ then. And the condition that $x,y \neq 1$ might be as well weakened here to the more involved: either $x \neq 1$ and $y = 0$, or *vice versa*.

For proof we start with the case where the elements x and y are dimensionally *alien*, i.e. such that $D(x) \cap D(y) = \varnothing$. The other case boils then easily down to the former.

Indeed, suppose $x,y \neq 1$ and they are dimensionally alien. As $x \neq 1$, all its atoms must be of different dimensions; and similarly for y. By finite atomicity of L, both $At(x)$ and $At(y)$ are finite; hence the union $At(x) \cup At(y)$ is a finite set of atoms too, and by alienage its elements are all of different dimensions again. Thus by throughout orthogonality: $\sup(At(x) \cup At(y)) \neq 1$. But in any lattice (cf. [4], p.8, Exercise 33, the conditions of which have been clearly satisfied here) we have: $\sup(At(x) \cup At(y)) = \sup At(x) \vee \sup At(y)$; and in any atomistic one the latter is equal to $x \vee y$. Thus $x \vee y \neq 1$, and so in transposed form the lemma has been established for alien elements.

And if the elements x,y are not alien, their join $x \vee y = \sup(At(x) \cup At(x))$ may be viewed as one of three elements alien to one another: $x \vee y = (x_1 \vee z) \vee y_1$, with $x_1 = \sup(At(x) - At(y))$, $z = \sup(At(x) \cap At(y))$, and $y_1 = \sup(At(y) - At(x))$. And we simply repeat the foregoing proof first for the elements x_1 and z, and then for $x_1 \vee z$ and y_1. QED.

Now at last we are ready to state the relationship of $\boldsymbol{R}$ to $\boldsymbol{R}'$ – rather simple and obvious, to be sure, only its stringent conditions are less so. Here it is:

(13) Let the two semantic frames F and F' be both Humean. And let both their respective ontologies $(SE, \boldsymbol{R})$ and $(SE, \boldsymbol{R}')$ be finitely atomistic and dimensionally determinate, with their respective

collections of dimensions $\boldsymbol{D}$ and $\boldsymbol{D}'$ orthogonal throughout. If under these conditions F' is a conjunctive extension of F, then $\boldsymbol{R} = \boldsymbol{R}'/SE$.

To see inclusion $\subset$ here, it will do to assume that SE is a subsemilattice of the join-semilattice with unit SE'. The inclusion means then that every maximal ideal of SE is the restriction to SE of some maximal ideal of SE'; i.e.,

$$\wedge R \in \boldsymbol{R} \vee R' \in \boldsymbol{R}' : R = R' \cap \mathrm{SE}.$$

Indeed, take any $R \in \boldsymbol{R}$. As $R \subset SE'$, and as its members are compossible in SE, they are also compossible in SE'; compossibility being preserved upwards under inclusion. Hence it is included in some maximal ideal of SE': $R \subset R'$, for some $R' \in \mathbf{R}'$. But this satisfies the conditions of lemma (11), and so we obtain the relevant identity therefrom simply by substitution.

As for the opposite inclusion, it means that restricting a maximal ideal of SE' to the sublattice SE always yields a maximal ideal of the latter. I.e.,

$$\wedge R' \in \boldsymbol{R}' : R' \cap \mathrm{SE} \in \boldsymbol{R}.$$

Indeed, but to show this we need three steps. Firstly observe that by lemma (6) the intersection $R = R' \cap \mathrm{SE}$ is an ideal of SE. Now suppose R is not maximal in SE. So – by lemma (8) – there must be an $y \in SE$ which is compossible with all members of R, but nevertheless is not in R; i.e., $x \vee y \neq \lambda$, for every $x \in R$, but $y \notin R$. Now secondly, and by lemma (8) again, as $y \notin R'$ and R' is maximal in SE', there must be some $z \in R'$ which is incompatible with y. And as y is compossible with all the members of R, that z must lie outside of R; i.e., $z \in R' - R$. (Cf. Figure 4.) And $y \vee z = \lambda$.

Thirdly, if y is incompatible with z, then by lemma (12) their atoms must clash on some common dimension $D' \in \boldsymbol{D}'$. That dimension cannot be any from $\boldsymbol{D}$, as – setting here $D(A) = \bigcup \{D(x): x \in A\}$ – we have $D(R) = \boldsymbol{D}$; and as y is compossible with every member of R. Consequently, the clash is due to a join of some atoms of the lattice SE' which is of the form $t_1 \vee t_2$, with $t_1 \in D_1'$, $t_2 \in D_2'$, and which in the sublattice SE appears as one of its atoms. I.e., we have: $(t_1 \vee t_2) \in D$, with $D \in \boldsymbol{D}$, its indivisibility in SE marked here by taking it redundantly in brackets; and $(t_1 \vee t_2) \leq y$, as one atom.

Thus $y \in X^+$, that set defined as in Section 3. So the element z clashes with y either on dimension D_1', or on dimension D_2', but not on both. For only one of these dimensions can be involved in the element z, as otherwise it would be in *SE*, hence in *R*. Say, it clashes on the former: for some $t \in D_1'$ we have: $t \leq z$ and $t \neq t_1$.

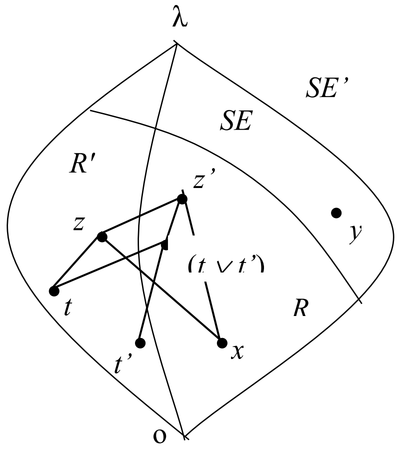

Figure 4

Thus $z = x \vee t$, for some $x \in X^-$, again defined as in Section 3. But *R'* contains also some atom $t' \in D_2'$, as $D(R') = \boldsymbol{D}'$. However, being an ideal and containing both *z* and *t'*, the set *R'* must also contain their join: $z' = z \vee t' = x \vee (t \vee t')$. In view of these identities we see – cf. Figure 4 again! – that *z'* belongs to the set X^+, hence it is a member of the sublattice *SE*, and so of *R*. Obviously, if $y \vee z = \lambda$ and $z \leq z'$, then $y \vee z' = \lambda$ too – which is in patent contradiction to the initial supposition of the element's *y* compossibility with all members of *R*, hence also with *z'*. QED.

And so in a very cumbersome way we have proved a very simple identity. Let someone else improve on it, if possible.

6. Disjunctive Extensions

We start again with an example. It is very much like that of Section 3: observing a tiny square and describing its appearance. This time, however, the observer finds out that the square in question takes only t h r e e hues: it turns either black, or grey, or white. Denoting as before the square itself by *X*, the observer is faced with three atomic situations which between them constitute a logical dimension $D = \{(X,B), (X,G), (X,W)\}$ of some unspecified semantic frame *F*.

After a time, however, he discovers that under closer inspection the intermediate hue of "grey" is not always quite the same at all. He was wrong: actually it comes in two shades, either as a "darker grey", or as a "lighter grey", which he has not noticed before as they differ just slightly. This discovery marks the transition to another semantic frame *F'*, equally unspecified, but with dimension $D' = \{(X,B), (X,dG), (X,lG), (X,W)\}$ in place of the former dimension *D*. Evidently, the dimension *D'* of *F'* somehow "extends" the dimension *D* of *F*, but how exactly are these two frames – and their respective ontologies $(SE',\boldsymbol{R}')$ and $(SE,\boldsymbol{R})$ – related to one another?

Clearly, the new way of extending a semantic frame is neither adjunctive, nor conjunctive, for the number of logical dimensions stays now the same – in contradistinction to the other two. Moreover, in the transition from $\boldsymbol{D}$ to $\boldsymbol{D}'$ some element of the lattice *SE disappears* altogether, and new ones take its place in the lattice *SE'*; and *vice versa*. In our example the atom of *SE* described by "*X* is grey" has been supplanted in *SE'* by two new ones "*X* is of lighter grey" and "*X* is of darker grey", but this time *in disjunction*. The proposition "*X* is grey" is equivalent in *F'* to the proposition "*X* is of lighter or darker grey", but no *single* element in the lattice *SE'* corresponds now to either of these propositions: the element (X,G) is in *SE*, but not in *SE'* any more. Thus *SE* cannot be a sublattice of *SE'*, of course.

We shall call this third way of extending a semantic frame "disjunctive extension".

Generally, we adopt the following conditional definition of it:

(14) Let *F* and *F'* be two Humean semantic frames; let their respective lattices *SE* and *SE'* be finitely and uniquely atomistic, dimensionally determinate, and with the dimensions in both $\boldsymbol{D}$ and $\boldsymbol{D}'$ orthogonal throughout.

Then we say: *SE'* is a *disjunctive extension* of *SE* if and only if there is a dimension $D \in \boldsymbol{D}$, and a dimension $D' \in \boldsymbol{D}'$, such that: $\boldsymbol{D}' - \{D\} = \boldsymbol{D}' - \{D'\}$; and if there is also a function $d: D' \rightarrow D$ which satisfies these conditions:

a) it is not one-to-one;

b) it is onto;

c) it can be extended to an epimorphism from *SE'* to *SE*.

To define that function means to be given a partition of dimension D' into blocks of elements *indistinguishable* within frame F, as shown in Figure 5, with at least one of the blocks non-trivial.

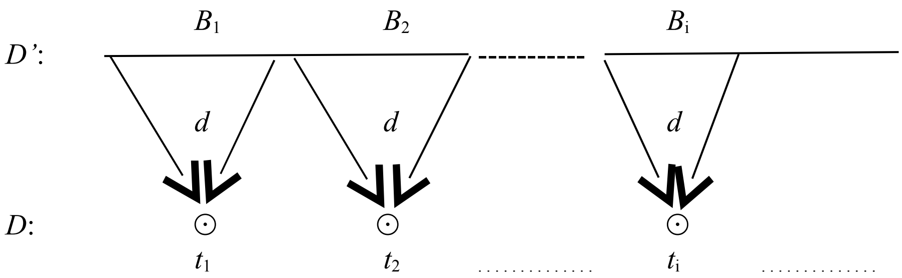

Figure 5

Denoting by $D'/d = \{B_1, B_2, \ldots, B_i, \ldots\}$ a partition of dimension D' of SE' like that hinted at in Figure 5, we have trivially, for any $t_1', t_2' \in D'$:

$$d(t_1') = d(t_2') \quad \text{iff} \quad \vee\, B_i \in D'/d\colon t_1', t_2' \in B_i.$$

Now we want to extend function $d\colon D' \to D$ to a $\{0,1\}$-homomorphism from SE' onto SE; i.e., to a function $h\colon SE' \to SE$ such that:

a) $SE = h/SE'/$
b) $h(o) = o$, and $h(\lambda) = \lambda$;
c) $h(x \vee y) = h(x) \vee h(y)$, for any $x,y \in SE'$.

By construction the join-semilattices SE and SE' are generated by their respective atoms $QA = \bigcup \boldsymbol{D}$ and $QA' = \bigcup \boldsymbol{D}'$:

$$SE = [QA] = \{x \in SE\colon x = \sup A, \text{ for some } A \in \text{Fin } QA\},$$
$$SE' = [QA'] = \{x' \in SE'\colon x' = \sup A', \text{ for some } A' \in \text{Fin } QA'\}.$$

But as they differ on one dimension only, i.e. as:

$$QA - D = QA' - D',$$

they have a common core X^{o} such that

$$\begin{aligned} X^{o} &= SE \cap SE' \\ &= \{x \in SE\colon \neg\, t \le x, \text{ for any } t \in D\} \cup \{\lambda\}, \\ &= \{x' \in SE'\colon \neg\, t' \le x', \text{ for any } t' \in D'\} \cup \{\lambda\}. \end{aligned}$$

Thus, as by construction both D and D' are orthogonal to the core X^{o}, we have:

(15) $SE = X^o \cup X^o \bullet D = X^o \bullet (\{o\} \cup D)$ $SE' = X^o \cup X^o \bullet D' = X^o \bullet (\{o\} \cup D')$.

Clearly, $SE' - X^o = X^o \bullet D' - \{\lambda\}$. Thus every element x' of the set $SE' - X^o$ is of the form $x' = x \vee t'$, for some $x \in X^o$ and some $t' \in D'$. And now we define the function h: $SE' \to SE$ as follows: for any $x' \in SE'$,

(16) $$h(x') = \begin{cases} x', \text{ if } x' \in X^o \\ x \vee d(t) \text{ - with } x' = x \vee t', x \in X^o, t' \in D' \text{ - if } x' \in SE' - X^o \end{cases}$$

Let us just see that indeed, for any $x', y' \in SE'$, we have then:

(17) $h(x' \vee y') = h(x') \vee h(y')$.

Observe that X^o is an ideal of the join-semilattice SE'. Thus if $x' \vee y' \in X^o$, then $x', y' \in X^o$. So $h(x' \vee y') = x' \vee y' = h(x') \vee h(y')$ all right.
Now suppose $x' \vee y' \notin X^o$. Then either x' is not there, or y', or both. Let us consider just the last case, which is the harder one: $x', y' \in SE' - X^o$. If $x' \vee y' = \lambda$, then $x' \vee y' \in X^o$, and we get $h(x' \vee y') = x' \vee y' = \lambda$, all right. Suppose therefore that $x' \vee y' \neq \lambda$. Then we have, for some $x,y \in X^o$ and some $t_1', t_2' \in D'$: $x' = x \vee t_1'$ and $y' = y \vee t_2'$. Consequently, what eventually we have got to prove is this:

$$h((x \vee t_1') \vee (y' \vee t_2')) = h((x \vee y) \vee (t_1' \vee t_2'))$$
$$= h(x \vee t_1') \vee h(y \vee t_2').$$

The first equality is patent. And for the second observe that as $x' \vee y' \neq \lambda$, so $t_1' \vee t_2' \neq \lambda$, of course. But as t_1', t_2' are of the same dimension, their compossibility means they are identical: $t_1' = t_2' = t'$. Thus, in view of definition (16) we are left with the equalities:

$$h(x' \vee y') = h((x \vee y) \vee t')$$
$$= (x \vee y) \vee d(t'),$$

on the one hand. And on the other we have:

$h(x') = h(x \vee t') = x \vee d(t')$,
and $h(y') = h(y \vee t') = y \vee d(t')$;
hence $h(x') \vee h(y') = (x \vee y) \vee d(t')$,

which is exactly what we were looking for. QED.

Under disjunctive extension the relationship between ***R*** and ***R'*** is very simple. For solely by pure function theory we have got there:

(18) Let *SE*, *SE'* be two arbitrary bounded join-semilattices, with ***R***, ***R'*** their respective collections of all maximal ideals, and the function

> h: $SE' \rightarrow SE$ an epimorphism. Setting then $h/\boldsymbol{R}'/ = \{h/R'/: R' \in \boldsymbol{R}'\}$, we have: $\boldsymbol{R} = h/\boldsymbol{R}'/$.

I.e., under the mapping h the images of the maximal ideals in SE' are the maximal ideals in SE.

Indeed, inclusion $\supset$ means: $\wedge\, R' \subset \boldsymbol{R}'$: $h/R'/ \in \boldsymbol{R}$. So let us take an arbitrary $R' \in \boldsymbol{R}'$. Then $I_o = h(R')$ is a proper ideal of SE (cf. [3], p. 100). Consider now an arbitrary ideal I_1 of SE such that $I_o \subset I_1$. Then $h^{-1}(I_o) \subset h^{-1}(I_1)$. (Cf. [5], p. 81, proposition (23)). Moreover, $R' \subset h^{-1}(I_o)$. (Cf. *ibid*., prop. (25)). But as I_1 is an ideal of SE, so $h^{-1}(I_1)$ is one of SE', as h is *onto*. (Cf. [3], *ibid*.). And as $I_o \subset I_1$, we have: $h^{-1}(I_o) \subset h^{-1}(I_1)$. (Cf. [5], prop. (23) again). Thus $R' \subset h^{-1}(I_1)$. But both are ideals of SE', so with R' being a maximal one there, we get: either $R' = h^{-1}(I_1)$, or $h^{-1}(I_1) = SE'$. If the latter, then $h(h^{-1}(I_1)) = h(SE') = SE$, as h is onto. But $h(h^{-1}(I_1)) = I_1$. (Cf. [5], prop. (24)). Hence $I_1 = SE$, showing that the ideal $I_o = h(R')$ is maximal in SE. (Cf. Figure 6.)

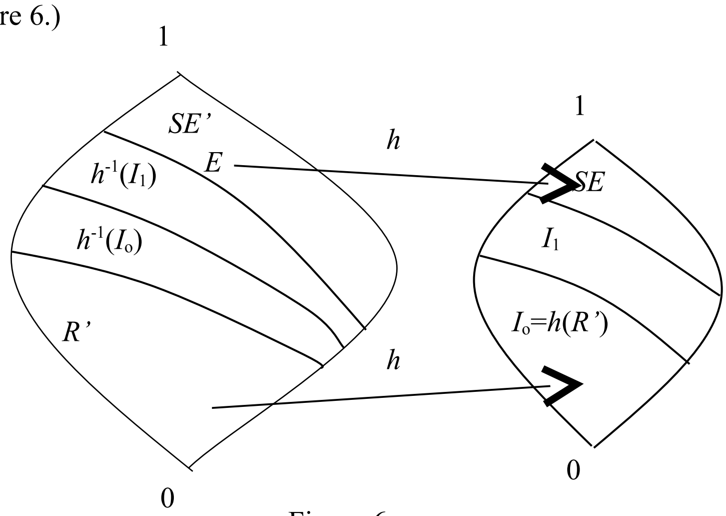

Figure 6

And the opposite inclusion $\subset$ means: $\wedge\, R \in \boldsymbol{R}$: $h^{-1}(R) \in \boldsymbol{R}'$. So let us take any $R \in \boldsymbol{R}$, setting $I_o' = h^{-1}(R)$. As R is a proper ideal of SE, and h is onto, I_o' must be one of SE'. Suppose now there is another proper ideal I_1' of SE' such that $I_o' \subset I_1'$. (Cf. Figure 7.) Then we have: $h(I_o') \subset h(I_1')$. But $h(I_o') = h(h^{-1}(R)) = R$ (cf. [5], prop. (24) again). Thus $R \subset h(I_1')$, which is the clinching inclusion. For being the image of a proper ideal of SE', the set $h(I_1')$ is also one of SE. And as R is maximal there, we get: $R = h(I_1')$. Thus $h^{-1}(R) = h^{-1}(h(I_1'))$. But we have always (cf. [5], prop. (25) again): $I_1' \subset h^{-1}$

($h(I_1')$). So eventually $I_1' \subset h^{-1}(R)$, i.e. $I_1' \subset I_o'$. This means $I_1' = I_o'$, showing I_o' to be maximal in SE': $h^{-1}(R) \in \boldsymbol{R}'$. QED.

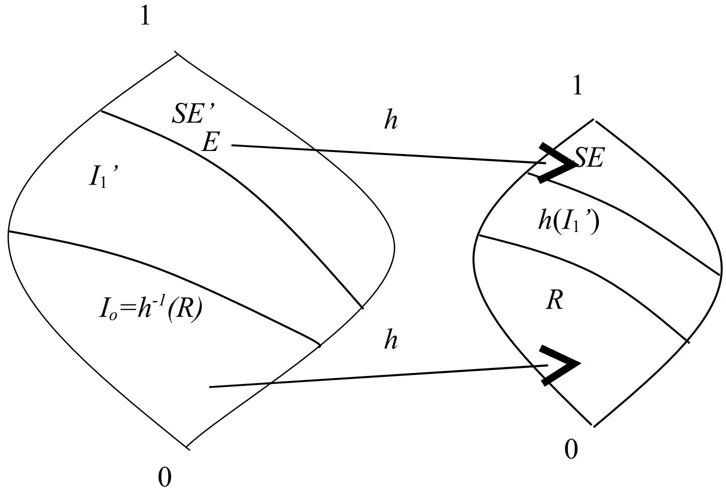

Figure 7

7. Sem**antical Equivalence**

For disjunctive extension crucial, of course, is the partition D'/d; i.e., the correlation between blocks of D'/d in SE' and elements of dimension D in SE. How is it being established? Well, in a very roundabout way, for in reference to the frames F and F' in their entirety – not, as so far, merely to their ontologies ($SE,\boldsymbol{R}$) and ($SE',\boldsymbol{R}'$). In particular we have to refer to their languages L and L', which for just that purpose we have assumed (cf. [2], Section 5) as staying unchanged under atomic extensions: $L = L'$.

Let us sketch out the main point of that reference, for disjunctive extensions only.

Take any $t_1', t_2' \in D'$, with D' being the disjunctively extending dimension in $\boldsymbol{D}'$, as before in Section 6. The condition for those atoms to be in one block of the partition D'/d is this:

(19) $\vee B_i \in D'/d$: $t_1', t_2' \in B_i$ iff $\wedge \alpha \in L$: $t_1' \in V(\alpha)$ iff $t_2' \in V(\alpha)$,

where $V(\alpha)$ is the set of elementary situations each of which *verifies* propositions α. (Cf. [1], Section 1, def. (9).) Thus in the frame F' – let alone in F – those atoms are indistinguishable in so far as there is no proposition in the language $L' = L$ of that frame to differentiate them. In F' the difference between them is “ineffable”.

Thus the dimensions D' of F' and D of F are correlated as follows. Let $V(a)$ be all the verifiers of proposition α as defined in frame F, and $V'(\alpha)$ – as defined in F'. Then we have, for any atom $t \in D$, and for any block of atoms $B \in D'/d$:

(20) $B = d^{-1}(t)$ iff $\wedge\, \alpha \in L: t \in V(\alpha)$ iff $\vee\, t' \in B: t' \in V'(\alpha)$.

Then we say that the block of atoms B in frame F' is *semantically equivalent* to the single atom t in frame F.

8. Final Remark

As stated in [1] and [2] a semantic frame F is an algebraic system consisting of three couples: an abstract logic (L,Cn), where L is an algebra of formulae (a “language”), and Cn is the consequence operation on L (cf. [6]); an ontology $(SE,\boldsymbol{R})$ as characterized above; and a semantic (Z,Z_o), where Z is a function of the form $Z: \boldsymbol{R} \rightarrow P(L)$ mapping realizations into sets of propositions complete under Cn, and Z_o is the totality of truths in L. (The abstract logic (L,Cn) turns “concrete” as soon as Cn gets coupled axiomatically with Z_o.)

Semantic frames can hardly be extended with regard to all their constituents at once: all points of reference get lost under such radical extension. Therefore we have confined our investigation of extensions from frame F to frame F' to their ontologies only, keeping the rest constant: $L' = L$, $Cn' = Cn$, $Z'/\boldsymbol{R}'/ = Z/\boldsymbol{R}/$, and $Z_o' = Z_o$. Thus only the purely *ontological* extensions of semantic frames have been taken in consideration.

Moreover, we have investigated them under extremely tight conditions, and so it suggests itself to loosen them for the sake of generality. For instance, to consider frames not necessarily Humean, i.e. such where two elementary situations may go always together without being identified with each other thereby. Or non-atomistic frames, in which – rather plausibly – possible worlds are “bottomless”, and all would-be “atomic” situations are so just “until further notice”. Or non-orthogonal ones, in which – quite plausibly again – atoms of different dimensions may clash with one another, like temperature and moisture in thermodynamics. Or ones without a comprehensive dimensional determination at all – and so on.

We are happy to admit the point of such generalized frames. We even have tried to lay down a groundwork for such inquiries, as we started – in defining the notion of a "semantic frame" – from the weakest axiomatic base possible, restricting it then step by step with additional axioms. To our opinion, however, the heaviest limitation has been the *quantitative* one. Of the three ways to extend a semantic frame – adjunctively, conjunctively, or disjunctively – we have taken account of such variants only where the *numbers* involved are at the lowest. Thus we have considered extensions of *SE* to *SE'* where the one arises from the other either by *one* dimension being added to it; or by *one* dimension being split in *two*; or by *one* dimension just being blown up a bit. (And under the assumption of their orthogonality to the rest of them.)

Moreover, we have assumed the frames to be *finitely* atomistic, and we have defined "orthogonality" with reference to *finite* joins of atoms.

All this needs more generality, with possibly infinite cardinalities; and weaker assumptions. There is a lot of open problems left here.

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