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EXTENDING ATOMISTIC FRAMES

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Abstract
A "semantic frame" is bounded join-semilattice of *elementary situations*, with its maximal ideals to represent possible worlds and mapped into the complete sets of propositions determined by a given abstract logic (L, Cn). A frame is *Humean* if the elementary situations are separated by its possible worlds, and it is *atomistic* if the semilattice is so. One frame is the *extension* of another if the latter is an {0,1}-subsemilattice of the former satisfying certain conditions discussed.

1. Recapitulation of [1]

As in [1], a "semantic frame" is to be a sextuple of the form ((L, Cn), (SE, $\boldsymbol{R}$), (Z, Z_0)), where (L, Cn) is an abstract logic, SE is the set of "elementary situations", the members of $\boldsymbol{R}$ are realizations ("possible worlds"), Z is a function of the form $\boldsymbol{R} \to P(L)$, and Z_0 is the set of all true propositions of L. That sextuple has to satisfy the following nine conditions:

(L1) (L, Cn) is classic.

This is to say (cf. [2], p. 44) that the consequence operation is finite, i.e. for any $X \subset L$:

(L1.1) $Cn\, X = \bigcup\{Cn\, Y: Y \in \mathrm{Fin}\, X\}$;

and that there are in the language L two operations, say $n(\alpha)$ and $d(\alpha,\beta)$, characterized for any $\alpha, \beta \in L$ as follows:

(L1.2) $\alpha \in Cn\, X$ iff $Cn\, (X, n(\alpha))=L$

(L1.3) $Cn(X, d(\alpha,\beta)) = Cn(X, \alpha) \cap Cn(X, \beta)$.

Observe that being classic, the abstract logic (L, Cn) contains maximally consistent sets, i.e. theories complete under Cn. We denote their totality by $\boldsymbol{Z}$.

The remaining conditions are these:

(R1) $\boldsymbol{R} \subset P(SE)$
(R2) $\boldsymbol{R} \neq \varnothing$
(R3) $\cup \boldsymbol{R} \neq SE$
(R4) $\cap \boldsymbol{R} \neq \varnothing$
(R5) $\bigwedge R_1, R_2 \in \boldsymbol{R}: R_1 \subset R_2 \Rightarrow R_1 = R_2$;

and setting

$$V(\alpha) = \{x \in SE: \bigwedge R \in \boldsymbol{R}: (x \in R \Rightarrow \alpha \in Z(R)\},$$

where the members of $V(\alpha)$ are the "verifiers" of the proposition α, we assume:

(Z1) $Z/\boldsymbol{R}/ \subset \boldsymbol{Z}$
(Z2) $Z_0 \in Z/\boldsymbol{R}/$
(Z3) $\bigwedge \alpha \in L, R \in \boldsymbol{R}: \alpha \in Z(R) \Rightarrow \bigvee x \in R: x \in V(\alpha)$.

Adding further conditions we obtain semantic frames of a special kind. In particular we shall assume the following two. Setting, for any $z \in SE$, $A \subset SE$:

(1) $K(z,A)$ iff $\bigwedge R \in \boldsymbol{R}: z \in R$ iff $A \subset R$,

we call a frame *conjunctive* if:

(R6) $\bigwedge A \in Fin\, SE \ \bigvee z \in SE: K(z,A)$.

And we call it *R-compact* if:

(R7) $\bigwedge A \subset SE\ (\bigwedge B \in Fin\, A\ \bigvee R \in \boldsymbol{R}: B \subset R) \Rightarrow \bigvee R' \in \boldsymbol{R}: A \subset R'$).

Thus only conjunctive and R-compact frames will be taken into consideration henceforth.

2. Elementary situations factorized

Setting $r(x) = \{R \in \boldsymbol{R}\colon x \in R\}$, and $x \vdash y$ iff $r(x) \subset r(y)$ — read "x entails y" — we obtain a quasi-ordering $(SE, \vdash)$ of elementary situations. Setting then $x \sim_r y$ iff $r(x) = r(y)$, and factorizing SE by that "r-equivalence", we arrive as usual at the partial ordering $(SE/r, \leq)$, here with $/x/ \leq /y/$ iff $y \vdash x$. Clearly, sets of the form $/x/_r$ are blocks of mutually *inseparable* elementary situations.

As we have shown in [3], the following two propositions hold for the factor-set SE/r:

(1) Under (R6), the partial ordering $(SE/r, \leq)$ is a join-semilattice.

And secondly:

(2) Under (R7), collections of the form $R/r = \{/x/ \in SE/r\colon x \in R\}$, for all $R \in \mathbf{R}$, are the maximal ideals of the semilattice SE/r.

Moreover, by construction the semilattice SE/r is *separated* by its maximal ideals, i.e., for any $x, y \in SE$ we have: $/x/ \neq /y/ \Rightarrow \vee R \in \boldsymbol{R}\colon\ /x/ \in R/r$ and $/y/ \notin R/r$, or conversely.

For all $z \in SE$ we also have:

(3) $\bigwedge A \in Fin\ SE$: $/z/ = \sup\{/x/\colon x \in A\}$ iff $K(z, A)$.

Let us show it just for the case $A = \{x, y\}$:

(3') $/z/ = \sup\{/x/, /y/\}$ iff $K(z, \{x, y\})$.

Indeed, by definition we have: $z \in R$ iff $R \in r(z)$, and $/x/ = /y/$ iff $r(x) = r(y)$, for all $x, y, z \in SE$, $R \in \boldsymbol{R}$. Consequently,

(4) The join-semilattice $(SE/r, \leq)$ is anti-isomorphic to the meet-semilattice $(r/SE/, \subset)$.

Hence we get: $/z/ = \sup\{/x/, /y/\}$ iff $r(z) = r(x) \cap r(y)$. On the other hand, $K(z, \{x, y\})$ is to say that $z \in R$ iff $x, y \in R$, for all $R \in \boldsymbol{R}$; or equivalently: $R \in r(z)$ iff $R \in r(x)\ .\ R \in r(y)$, i.e. iff $R \in r(x) \cap r(y)$. QED.

Obviously, the sets $\Omega = \cap\boldsymbol{R}$ and Λ (lambda) $= SE - \cup\boldsymbol{R}$ are r-equivalence classes. In SE/r the former is the zero, and the latter is the unit: $\Omega \leq /x/ \leq \Lambda$, for any $x \in SE$. Members of Ω are *necessary* situations, members of Λ are the *impossible* ones; the rest $\cup\boldsymbol{R} - \cap\boldsymbol{R}$, if any, are *contingent.* ("If any", for the conditions (R1)-(R7) leave that open.)

3. Atomistic Frames

For an arbitrary quasi-ordering (SE, $\vdash$) we define the notion of a *quasi-atom* (or "atomic situation", if SE is part of a semantic frame), with QA to be their totality: for any $x \in SE$,

(1) $$x \in QA \text{ iff } x \notin \Omega \text{ and } \bigwedge y \notin \Omega: x \vdash y \Rightarrow y \vdash x.$$

Thus entailments between quasi-atoms, if any, are always reciprocal. And for any $x \in SE$, $A \subset SE$ we set:

(2) $$At(x) = \{y \in QA: x \vdash y\},\ At(A) = \bigcup \{At(x): x \in A\}.$$

For the semantic frames to be considered here we shall stipulate some conditions of "atomicity". To begin with, let us adopt some definitions.

Let SE be again an arbitrary quasi-ordering, and let x be an arbitrary member of it. We shall say that SE is:

atomic iff $x \notin \Omega \Rightarrow At(x) \neq \varnothing$;
finitely atomic iff atomic and $x \notin \Lambda \Rightarrow At(x) \in Fin\ QA$;
atomically determinate iff atomic and $At(x) \subset R \Rightarrow x \in R$,
(3) for any $R \in \boldsymbol{R}$;
atomistic iff atomic and atomically determinate;
finitely atomistic iff finitely atomic and atomically determinate.

When SE is part of a semantic frame, the same terms will be used of the whole frame, too.

For the following we adopt these conditions of "atomicity":

(R8) SE is finitely atomic.
(R9) SE is atomically determinate.

Henceforth we shall consider semantic frames only which are finitely atomistic.

In view of (R9) we get immediately:

(4) $$\bigwedge x \in SE: K(x, At(x)),$$

as $x \in R \Rightarrow At(x) \subset R$ simply by definition. Consequently,

(5) $\bigwedge R_1, R_2 \in \boldsymbol{R}: At(R_1) = At(R_2) \Rightarrow R_1 = R_2$.

Let us take any $x \in R_1$, then $At(x) \subset R_1$ by (4); and so $At(x) \subset R_2$ by hypothesis. Hence $x \in R_2$ by (4) again. And similarly the other way round.

Implication (5) is condition "(A2)" stipulated in [1]. And condition "(A4)" there: $\bigwedge z \in SE \bigvee A \in Fin\ QA$: $K(z,A)$ follows trivially from (4). (Observe, however, that in an atomically determinate frame that A need not be unique.)

Let us note some relationships following directly from the definition of a quasi-atom (1). Firstly,

(6) $x \in QA \wedge x \vdash y \Rightarrow \bigwedge z: z \notin \Omega \Rightarrow (y \vdash z \Rightarrow z \vdash y)$.

For by $x \in QA$ and (1) we have: $z \notin \Omega \Rightarrow (x \vdash z \Rightarrow z \vdash x)$. Thus, in view of $z \notin \Omega$, the consequent implication holds. But $x \vdash y$ and $y \vdash z$, by supposition, so $x \vdash z$. Hence $z \vdash x$, in view of the foregoing. And as by supposition $x \vdash y$, we get $z \vdash y$ all right.

Secondly,

(7) $x \in QA \wedge x \vdash y \Rightarrow (y \notin \Omega \Rightarrow y \in QA)$,

which is obvious in view of (6).

Thirdly,

(8) $x \in At(z)$ iff $/x/ \subset At(z)$.

Implication $\Leftarrow$ is obvious. Conversely, suppose $y \in /x/$. By hypothesis $x \in QA$ and $z \vdash x$. By supposition $y \vdash x$, so $y \notin \Omega$, as x is a quasi-atom. Thus all three conditions in (7) have been satisfied, i.e. $y \in QA$. And as $x \vdash y$, again by supposition, we get $z \vdash y$, by transitivity. Consequently, $y \in At(z)$.

Now let $At(SE/r)$ be the lattice-theoretic atoms of the partial ordering $(SE/r, \leq, \Omega)$, the set Ω obviously being its zero. We have then by (1):

(9) $x \in QA$ iff $/x/ \in At(SE/r)$.

Indeed, $/x/ \in At(SE/r)$

iff $/x/ \neq \Omega$ and $(/y/ \leq /x/ \Rightarrow (/y/ = \Omega$ or $/x/ = /y/))$
iff $x \notin \Omega$ and $(x \vdash y \Rightarrow (y \in \Omega$ or $x \sim_r y))$
iff $x \notin \Omega$ and $(y \notin \Omega \Rightarrow (x \vdash y \Rightarrow y \vdash x))$
iff $x \in QA$.

Consequently, for any $x, y \in SE$ we have:

(10) $y \in At(x)$ iff $/y/ \in At_r(/x/)$.

Indeed, $y \in At(x)$ iff $y \in QA$ and $x \vdash y$
iff $/y/ \in At\,(SE/r)$ and $/y/ \leq /x/$, by (9)
iff $/y/ \in At_r(/x/)$.

(By "$At_r(/x/)$" we mean, of course, the collection of r-blocks $\{/y/ \in At\,(SE/r)\colon /y/ \leq /x/\}$.)

The following propositions hold:

(11) Under (R8), the partial ordering $(SE/r, \leq, \Omega, \Lambda)$ is finitely atomic.

Indeed, "atomic" predicated of a bounded partial ordering means that each of its non-zero elements contains some atoms: $x \notin \Omega \Rightarrow At_r(/x/) \neq \varnothing$; and "finitely" means here that each non-unit element contains only a finite number of them: $/x/ \neq \Lambda \Rightarrow At_r(/x/) \in Fin\,At(SE/r)$. As for the former, assume the antecedent: $/x/ \neq \Omega$. Thus $x \notin \Omega$. Hence, by (R8), $At(x) \neq \varnothing$; i.e., $x \vdash y$, for some $y \in QA$. In view of (9), this is equivalent to: $/y/ \leq /x/$, for some $/y/ \in At(SE/r)$. Consequently, $At_r(/x/) \neq \varnothing$. And as for the latter, assume the antecedent: $/x/ \neq \Lambda$. Thus $x \notin \Lambda$, so $At(x) \in Fin\,QA$, by (R8) again. Hence we can set: $At(x) = \{y_1, \ldots, y_n\}$, for some natural number n. By (10) this means that $At_r(/x/) = \{/y_1/, \ldots, /y_n/\}$, i.e. that it is finite.

Moreover, we have

(12) Under (R8) and (R9), SE/r is atomistic as a join-semilattice.

Indeed, "atomistic" as predicated of a join-semilattice means that for every $/x/$ there is an $\boldsymbol{A} \subset At(SE/r)$ such that $/x/ = \sup \boldsymbol{A}$. And in fact, substituting $A/At(x)$ in (2.3) we get, as under (R8) the set $At(x)$ is finite:

$$K(x, At(x)) \text{ iff } /x/ = \sup\{/y/\colon y \in At\,(x)\}.$$

In view of (R9) and (3.4) the left-hand part of that equivalence is a thesis, so the other one is, too. Hence, by (10), we get: $/x/ = \sup\{/y/\colon y \in At\,(x)\}$, which is just what is wanted.

In [3], p. 308, we have pointed out that a join-semilattice which is both atomistic and finitely atomic is a lattice, the meet of two elements being the supremum of the intersection of their respective sets of atoms. Thus we get eventually:

(13) Under (R8) and (R9), the partial ordering (SE/r, $\leq$) is a lattice, both atomistic and finitely atomic, with the join as in (2.3'), and the meet $/x/ \wedge /y/ = \sup (At_r(/x/) \cap At_r(/y/))$, for any $x, y \in SE$.

Observe that in view of (9) the atoms of the lattice SE/r coincide with r-blocks of quasi-atoms of the quasi-ordering SE. I.e.,

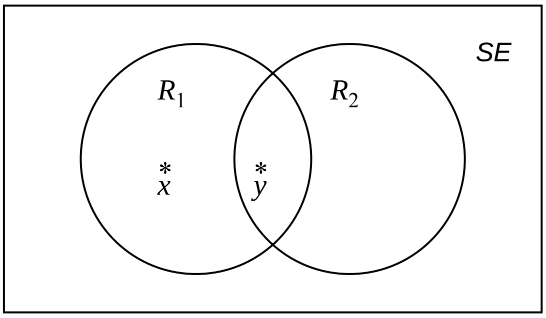
$$At(SE/r) = \{/x/ \in SE/r\colon x \in QA\} = QA/r.$$

4. Humean Frames

The generality of this investigation will be severely curtailed henceforth, for we shall limit it to one particularly simple and perspicuous kind of semantic frame only. Its ontology (SE, $\boldsymbol{R}$) is characterized by the condition:

(H) $\bigwedge x, y \in SE\colon\ r(x) = r(y) \Rightarrow x = y.$

Thus any two distinct elementary situations are *separable* by some realization, though not necessarily both ways. For if $x \vdash y$, then — as shown in diagram 1 — there is no $R \in \boldsymbol{R}$ to contain x without containing also y. Only reciprocal entailments are ruled out.



with $\boldsymbol{R} = \{R_1, R_2\}$.

Diagram 1

Frames satisfying (H) shall be called "Humean", for (H) is the formal analogue of a principle which constitutes the corner-stone of the metaphysical system presented in Hume's "Treatise". Most succinctly it may be stated thus: *whatever is distinct, is separable*. In formulations varying just slightly, this principle is explicitly invoked at least fourteen times (cf. [4], pp. 10,18, 24, 27, 36, 40, 54, 66, 79-80, 233, 259, 260, 632, 643). Oddly enough, despite its paramount position in the "Treatise", Hume's remarkable principle went almost unnoticed in the vast literature on him. (His famous doctrine that there is no necessary connexion between cause and effect appears as a straightforward corollary of that principle: the cause is distinct from the effect, so the latter is separable from the former.) Let us just point out here that in Selby-Bigge's otherwise excellent index of subjects, there is no entry on "separability". And in the entry "perception" only separability of perceptions from the mind is taken into account, the crucial one between them mutually is not even mentioned.

In Hume the quantifier "whatever" refers in his principle to *perceptions* (i.e. impressions or ideas): whatever perceptions are distinct, are separable. They are his elementary situations — his interpretation for the universe of discourse *SE*. And his realizations, i.e. his interpretation for the members of ***R***, are clearly *minds*, defined by him (p. 207) as "heaps or collections of different perceptions". Stepping beyond Hume's text, but not beyond his system, we have merely to add that those "heaps or collections" should be *maximal possible* ones; in particular no such heap should be a proper part of another. As realizations, Humean minds are complete; and obviously one mind cannot be part of another.

In a Humean frame all blocks of the partition SE/r are trivial: for any $x \in SE$,

(1) $/x/ = \{x\}$; i.e., $SE/r = SE/=$.

In particular, Ω and Λ are unit sets then: $\Omega=\{o\}$, $\Lambda=\{\lambda\}$, for some definite $o, \lambda \in SE$. There are thus in *SE* just one elementary situation that is necessary, and one that is impossible.

Moreover, in a Humean frame the structure $(SE, \vdash)$ is a partial ordering, and the relation $K(x, A)$ is a function from *Fin SE* to *SE*: x is the supremum of the finite set A under the partial ordering $\vdash$.

Consequently, *SE* is isomorphic to its partition SE/r:

(2) $(SE, \vdash) \cong (SE/r, \geq)$.

Thus, under all the foregoing conditions (R1)-(R9), we get the following proposition:

(3) In a Humean frame, the structure $(SE, \vdash, o, \lambda)$ is a bounded lattice which is both atomistic and finitely atomic. And ***R*** are its maximal

ideals.

I.e., for any $x, y \in SE$ there is an $z \in SE$ such that $z = x \vee y$ (namely iff $K(z, \{x,y\})$), and an $u \in SE$ such that $u = x \wedge y$ (namely iff $K(u, A)$, where $A = At(x) \cap At(y)$,).

5. Atomic Extensions

Let F, F' be two arbitrary semantic frames:

$$F = ((L, Cn), (SE, \boldsymbol{R}), (Z, Z_0))$$
$$F' = ((L', Cn'), (SE', \boldsymbol{R'}), (Z', Z_0')).$$

In [3] we considered F' to be an *extension* of F if the following held: Cn, Z_0 are just restrictions of Cn', Z_0' to L; the function Z' is subordinate to Z; and the ontology remains constant: $SE = SE'$, and $\boldsymbol{R} = \boldsymbol{R'}$. Here, however, we take an opposite stance.

Let F, F' be now two Humean frames, both atomistic and finitely atomic; and let QA, QA' be their respective sets of atoms. We shall say that the frame F' is an *atomic extension* of the frame F if its logic stays the same: $L' = L$, $Cn' = Cn$; its semantics is such that $Z'/\boldsymbol{R'}/ = Z/\boldsymbol{R}/$, $Z_0' = Z_0$; its ontology is such that $QA' = (QA - X) \cup Y$, where $X \subset QA$, $Y \subset \overline{QA}$, with X possibly empty; but if it is not, then neither is Y; and for an arbitrary $x \in QA$ we have one of the following three cases:

either a) $x \in QA'$;
or b) $x \notin QA'$, but $x \in SE'$, and $\bigvee A' \in Fin\ QA'$: $x = \sup A'$;
or c) $x \notin SE'$, but $\bigvee A' \subset QA' \bigwedge \alpha \in L$: $x \in V(\alpha)$ iff $\bigwedge x' \in A'$: $x' \in V'(\alpha)$,

the functions V, V' defined within their respective frames as in § 1.

Observe that in case (c) the set of A' takes in frame F' the place which was held in frame F by the atom x; and that the identity of place is determined here semantically by the totality of propositions verified by them:

$$\{\alpha \in L: x \in V(\alpha)\} = \{\alpha \in L': x' \in V'(\alpha), \text{ for some } x' \in A'\}.$$

In this sense, the element x and the set A' may be said to be “semantically equivalent”.

Observe also that in the transition from the frame F to a frame F' the element x, i.e. an atom of F, behaves as follows:

in case (a) x stays an atom also in F';
in case (b) x ceases in F' to be an atom, but reappears there as a compound;
in case (c) x disappears in F' altogether, but is replaced by a semantically equivalent set of atoms A'.

6. Adjunctive Extensions

Our abstract characterization of atomic extensions is rather opaque. To make it more perspicuous let us adopt two more assumptions. Firstly, we assume that, as in [1]:

(R10) Both F and F' are dimensionally determined.

This is to say that the respective sets of atoms QA and QA' are both partitioned into collections $\boldsymbol{D}$ and $\boldsymbol{D'}$ of *logical dimensions*, i.e. so that each of the blocks is a set of atoms which is both *exclusive* and *transverse*. By the former we mean that all atoms of one dimension exclude each other: $x \neq y \Rightarrow (x \in R \Rightarrow y \notin R)$, for any $x, y \in D$, and $D \in \boldsymbol{D}$, and any $R \in \boldsymbol{R}$; and similarly for the frame F'. By the latter we mean that each dimension intersects all realization of its frame: $D \cap R = \varnothing$, for any $D \in \boldsymbol{D}$, and any $R \in \boldsymbol{R}$. Immediately it follows that $D \cap R$ is always a unit set: $D \cap R = \{x\}$, for some $x \in QA$.

Secondly, and just for the sake of formal simplicity, let us assume that the two Humean frames in question are *uniquely atomistic*. This is to say (cf. [3], pp. 307-310) that on top of being atomistic the semilattices SE and SE' are such that for any $x \in SE$ (and SE'), and any $A \subset QA$ (and QA') we have:

$$x \neq \lambda \Rightarrow (x = \sup A \Rightarrow A = At(x)).$$

Thus we assume:

(R11) Both F and F' are uniquely atomistic.

(It is a separate question how to obtain unique atomicity for the partial ordering SE/r *via* conditions stipulated directly on the quasi-ordering SE itself.)

Let us note — and this will make (R11) somewhat less arbitrary — that for finitely atomistic lattices being uniquely atomistic is equivalent to being conditionally distributive, i.e. such that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $y \vee z \neq \lambda$; cf. [3], p. 310.

Case (a) will be called *adjunctive* extension. Under (R10) it consists in *QA'* differing from *QA* only in a new dimension *D'* having been adjoined to it: $QA' = QA \cup D'$, with $QA \cap D' = \varnothing$. In other words: $\boldsymbol{D} = \boldsymbol{D'} - \{D'\}$.

What is then the relationship between the two respective ontologies (*SE*, $\boldsymbol{R}$) and (*SE'*, $\boldsymbol{R'}$)?

As in group theory (cf. e.g. [5]), we shall call, for any $A, B \subset SE$, $A \bullet B = \{x \vee y \in SE: x \in A, y \in B\}$ the *product of subsets*. We have then:

(1) $$SE' = SE \bullet (D' \cup \{o\}) = SE \bullet D' \cup SE$$

as $SE \bullet \{o\} = SE$, and the product of subsets is easily seen to be distributive over their unions: $A \bullet (B \cup C) = A \bullet B \cup A \bullet C$.

Or conversely,

(2) $$SE = \{x \in SE': x = \lambda \text{ or } y \notin At(x), \text{ for any } y \in D'\}.$$

Thus $SE \subset SE'$. Moreover:

(3) If a Humean frame *F'* is an adjunctive extension of the Humean frame *F*, then under (R11) the set *SE* is a {0,1}-sublattice of *SE'*.

Indeed, take any $x, y \in SE'$. Supposing both are in *SE*, consider their join $z = x \vee y$. If $z = \lambda$, then obviously $z \in SE$. And if $z \neq \lambda$, then by (R11) — cf. [3], p. 309 — we have the identity: $At(x \vee y) = At(x) \cup At(y)$. As $x, y \in SE$, no atom of *D'* is contained in either $At(x)$, or $At(y)$. Hence none is in their unions either. Thus $x \vee y \in SE$, by (2). QED.

Now setting $x = \sup (At(x') - D')$, it might seem that the mapping $e: SE' \to SE$ such that

$$e(x') = \begin{cases} x', & \text{if } x' \in SE \\ x, & \text{if } x' \notin SE \end{cases}$$

is an endomorphism. It is readily seen, however, this it is not so. For take any $x', y' \in SE'$ such that $x' = x \vee z_1$, $y' = y \vee z_2$, for some $x, y \in SE$ such that $x \vee y \neq \lambda$; and for some $z_1, z_2 \in D'$ such that $z_1 \neq z_2$. Then, on the one hand, we have:

$$e(x' \vee y') = e(\lambda) = \lambda.$$

But, on the other:

$$\begin{aligned} e(x') \vee e(y') &= \sup(At(x') - D') \vee \sup(At(y') - D') \\ &= \sup((At(x) \cup At(z_1)) - D') \vee \sup((At(y) \cup At(z_2)) - D') \\ &= \sup At(x) \vee \sup At(y) \\ &= x \vee y \neq \lambda. \end{aligned}$$

Thus the mapping *e* as defined is not a homomorphism!

7. Ideals under Adjunctive Extension

What about the relationship between ***R*** and ***R'***? This is an intricate question, in need of a separate study, as under the mapping mentioned the sublattice *SE* = *e*/*SE'*/ is not an homomorphic image of the lattice *SE'*. At any rate, however, all the member-sets of ***R*** and ***R'***, respectively, have to be maximal ideals of the respective lattices *SE* and *SE'*. Let us see some of the obstacles when trying to obtain them from one another.

Starting from ***R***, we take an arbitrary $R \in$ ***R***, i.e. a definite maximal ideal of *SE*. And we fix one definite atom $t \in D'$. Thus we get a set-up as shown in diagram 2, all the relevant inclusions readily to be seen there. (The inner oval represents the set $QA' = QA \cup D'$, of course.) Somehow the couple (*R*, *t*) should determine a definite maximal ideal *R'* of *SE'*.

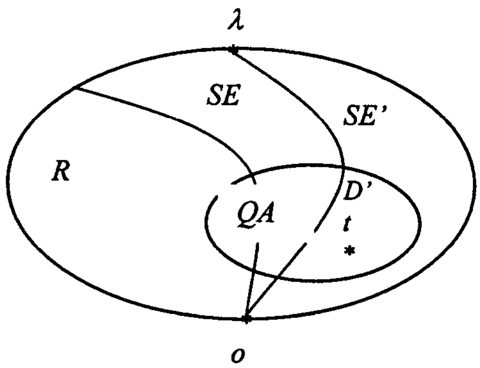


Diagram 2

The following construction comes immediately to mind. Consider the product $R \bullet \{t\} = \{x \vee t\text{: } x \in R\}$. Some members of R, however, may be incompatible in SE' with the new atom t. Let their totality, possibly empty, be the set $X(R, t) = \{x \in R\text{: } x \vee t = \lambda\}$. Clearly that set has to be withdrawn from the maximal ideal of SE' under construction. (Observe that the difference $R - X(R, t)$ is never empty, as always $o \in R$ and $o \notin X(R, t)$.) On the other hand, that ideal should contain all joins of the form $x \vee t$, with $x \in R$, except when $x \vee t = \lambda$; i.e. it should include the set $R \bullet \{t\} - \Lambda$. So consider the union:

$$Y = (R - X(R, t)) \cup (R \bullet \{t\} - \Lambda),$$

Is it an ideal of SE'? Not necessarily. It is easy to check, though somewhat tedious, that Y is closed downwards: if $x \in Y$, then $y \leq x$ implies $y \in Y$, for any $y \in SE'$. However, from $x', y' \in Y$ it does not follow that $x' \vee y'$ is in Y too. For suppose that x' belongs in Y to its first component, any y' to the second. Then we have: $x' \in R$, and $y' = y \vee t$, for some $y \in R$. Clearly, $x \vee y \neq \lambda$, but how are we to tell that the same goes for $(x \vee y) \vee t$, or even for $x \vee t$?

8. A Special Case

Now let us consider a rather special case of adjunctive extension, where the extending dimension D' is *orthogonal* to the sublattice SE if bereft of its unit λ. (Subsets A, B of a bounded lattice we call "orthogonal" to each other — cf. [3], p. 27 — if for every $x \in A$, $y \in B$: $x \vee y \neq \lambda$, $x \wedge y = o$.) Thus we set:

(*) $\quad \bigwedge x \in SE - \{\lambda\} \bigwedge t \in D'\text{: } x \vee t \neq \lambda,$

the condition $x \wedge y = o$ being satisfied automatically as $D' \cap QA = \varnothing$.

Then evidently $X(R, t) = \varnothing$, and $\lambda \notin R \bullet \{t\}$. Consequently,

(1) $\quad$ Under (*), $Y = R \cup R \bullet \{t\}$.

Moreover, the product of subsets $R \bullet \{t\}$ is then isomorphic to the direct product:

(2) $\quad$ Under (*), $R \bullet \{t\} \cong R \times \{t\}$,

and so also to R itself:

(3) Under (*), $R \bullet \{t\} \approx R$.

Clearly, the same holds of the whole lattice *SE*, i.e.,

(4) Under (*), $SE \cong SE \bullet \{t\}$,

the map e: $SE' \to SE$ as defined above establishing the one-to-one correspondence here.

Eventually we obtain a set-up as shown in diagram 3, where

$$SE \cup SE \bullet \{t\} = SE \bullet \{o, t\} \cong SE \times \{o, t\} = SE \times \{o\} \cup SE \times \{t\},$$

with "*t*" treated as a constant here. Turning it into a variable we get:

(5) $SE' = \bigcup\{SE \cup SE \bullet \{t\}: t \in D'\} = SE \cup \bigcup\{SE \bullet \{t\}: t \in D'\}$.

Observe that in the last formula the members of the union are all disjoint

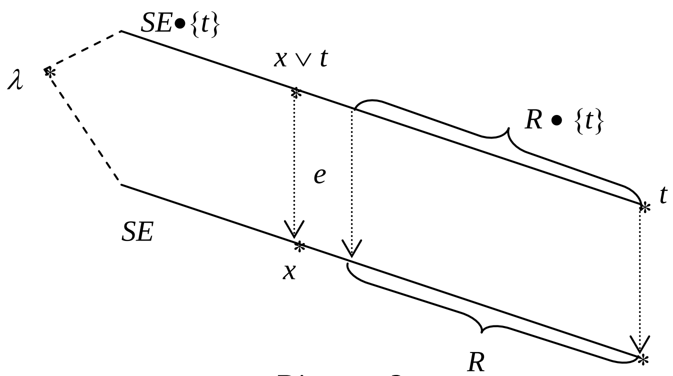


Diagram 3

except for λ which they have all in common.

For a $t \in D'$ let us call the product $SE \bullet \{t\}$ a sublattice of *SE' associated* with *SE*. Clearly, any two such sublattices are *disjoint up to* λ, i.e. for any t, $t' \in D'$ we have:

(5) $t \neq t' \Rightarrow SE \bullet \{t\} \cap SE \bullet \{t'\} = \{\lambda\}$.

For let the two relevant sublattices be T and T'. If $x' \in T \cap T'$, then $x' = x \vee t$, and $x' = y \vee t'$, for some $x, y \in SE$. Thus, adding sidewise, we get: $x' = (x \vee t) \vee (y \vee t') = (x \vee y) \vee (t \vee t') = (x \vee y) \vee \lambda = \lambda$.

Moreover, under (*) all the sublattices associated with *SE* are isomorphic, of course, as shown in diagram 4 below.

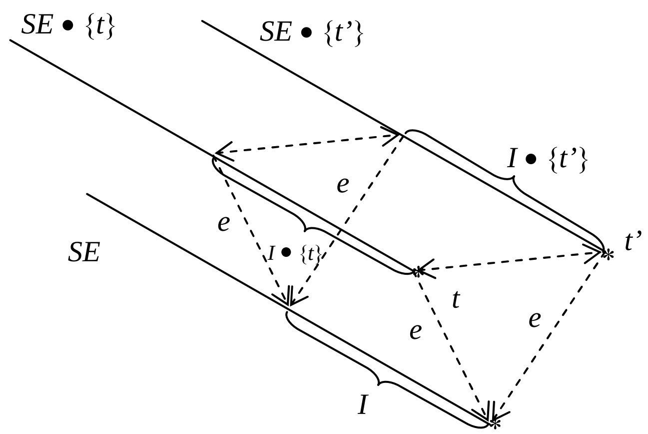


Diagram 4

Clearly,

(8) Every ideal of *SE* is also an ideal of *SE'*.

For let *I* be an ideal of *SE*, and take an arbitrary $y' \in SE'$. If $x \in I$, and $y' \leq x$, then $y' \in SE$ — as otherwise we should have $t \in At(x)$, a contradiction. So $y' \in I$. And the closure against joins is obvious.

Now let *SE'* be an adjunctive extension of *SE*. Then the following holds:

(9) If *I'* is a proper ideal of *SE'* then $I = I' \cap SE$ is one of *SE*.

The thesis means: $x, y \in I' \cap SE$ iff $x \vee y \in I' \cap SE$, for any $x, y \in SE'$. Implication $\Rightarrow$ holds for every sublattice of *SE'*, as then, by antecedent and *I'* being an ideal, we have: $x \vee y \in I'$; and $x \vee y \in SE$, as *SE* is a sublattice of *SE'*. Conversely, we get in the same way: $x, y \in I'$. And as $x \vee y \in SE$ by antecedent, we have: $t \notin At(x \vee y)$. Hence $t \notin At(x)$, $t \notin At(y)$. So $x, y \in SE$. Finally, as *I'* is proper, $\lambda \notin I'$; so $\lambda \notin I$.

Next let us prove a lemma:

(10) Under (*) the dimension *D'* being orthogonal to *SE*, and with *SE* finitely atomic (R8) and uniquely atomistic (R11): $\bigwedge x, y \in SE - \Lambda \ \bigwedge t \in D': x \vee t = y \vee t \Rightarrow x = y$.

For suppose: $x, y \in SE$; $x, y \neq \lambda$; $x \vee t = y \vee t$. Then $x \vee t, y \vee t \neq \lambda$, by (*) and the second. But $At(x \vee t) = At(y \vee t)$ by the third, so $At(x) \cup \{t\} = At(y) \cup \{t\}$, by R8 and R11, cf. [3], p. 309. Hence $At(x) = At(y)$ by algebra of sets, as $t \notin At(x), At(y)$. Thus $x = y$, as SE is atomistic by R11, cf. [3], p. 308.

Lemma (10) yields a theorem, easy to see, tedious to prove:

Under (*), R8, and R11; for any $I \subset SE$, and any $t \in D'$:
(11) if I is a proper ideal of SE, then $I \cup I \bullet \{t\}$ is one of SE'.

Set $T = I \bullet \{t\}$, for short. Then we have to show that for any $x', y' \in SE'$: $x', y' \in I \cup T$ iff $x' \vee y' \in I \cup T$. Observe, to begin with, that $\lambda \notin T$, as I is proper and (*) has been assumed.

For implication $\Rightarrow$ assume the antecedent. Then we have four cases at hand: ($x', y' \in I$) or ($x' \in I, y' \in T$) or ($x' \in T, y' \in I$) or ($x', y' \in T$). In case *one*, $x' \vee y' \in I$ trivially, as by (8) I is also an ideal of SE'. In case *two*, $y' = y \vee t$, for some $y \in I$. Thus $x' \vee y' = (x' \vee y') \vee t$; and as both x' and y are in I, we get: $x' \vee y \in I$. Consequently, $x' \vee y' \in T$. Case *three* is symmetric to the foregoing, and in case *four* the reasoning is essentially the same. Thus in all four cases we have: $x' \vee y' \in I \cup T$.

For implication $\Leftarrow$ assume again its antecedent. Then either $x' \vee y' \in I$, or $x' \vee y' \in T$. In case *one* we get $x', y' \in I$ immediately, as by (8) the set I is an ideal also of SE'. In case *two* we have: $x' \vee y' = z \vee t$, for some $z \in I$. So the atom t is contained in the join $x' \vee y'$, which means by unique atomicity (R11) that it must be contained in either x' or y'. Therefore, $x' \notin SE$, or $y' \notin SE$. Both are symmetric, so let us assume the former. Then we have: $x' = x \vee t$ and ($y' \in SE$ or $y' = y \vee t$), for some $x, y \in SE$. Take the first disjunct: $x' = x \vee t$, and $y' \in SE$. Thus $x' \vee y' = (x \vee y') \vee t$. Consequently, $z \vee t = (x \vee y') \vee t$, with both $z \in SE$, $x \vee y' \in SE$. (The latter as $x, y' \in SE$, and SE is a lattice.) So the conditions of lemma (10) have been satisfied, and we obtain therefrom: $z = x \vee y'$ (!). Thus $x, y' \in I$, as by assumption $z \in I$, and I is an ideal. But if $x \in I$, then $x \vee t \in T$. Hence $x', y' \in I \cup T$.

Under the second disjunct the reasoning is essentially the same again. (As $x' = x \vee t$, $y' = y \vee t$, we get: $(x \vee y) \vee t = z \vee t$. Hence $x \vee y =$ z by (10), yielding $x, y \in I$. So $x', y' \in T$.) QED.

In view of (5) and (6) we have clearly:

Every proper ideal I' of SE' is included in a sublattice

(12) of the form $SE \cup SE \bullet \{t\}$, with $t \in D'$. If $I' \not\subset SE$, I' is included in just one; otherwise in all.

This and lemma (11) suggest the following proposition on the relationship between the collections $\boldsymbol{R}$ and $\boldsymbol{R'}$ of the two frames in question:

Under (*), let the lattice SE' be an adjunctive extension of the dimensionally determinate Humean lattice SE. Then for any $R' \subset SE'$:

(13) $R' \in \boldsymbol{R'}$ iff $\bigvee R \in \boldsymbol{R} \bigvee t \in D'$: $R' = R \cup R \bullet \{t\}$, where $\boldsymbol{R}$ and $\boldsymbol{R'}$ are the maximal ideals of SE and of SE', respectively.

Suppose $R' \in \boldsymbol{R'}$. Then by (9) the set $R = R' \cap SE$ is a proper ideal of SE. So consider any other proper ideal I of SE such that $R \subset I$. Obviously $R \bullet \{t\} \subset I \bullet \{t\}$, for any $t \in D'$. But R is isomorphic to $R \bullet \{t\}$ by (3), and consequently $R \bullet \{t\}$ is a maximal ideal of the associated sublattice $SE_t = SE \bullet \{t\}$. As $I \bullet \{t\} \subset SE_t$, and I is a proper ideal of SE, we see in view of (4) that $I \bullet \{t\}$ is a proper ideal of SE_t. Hence $I = R$, i.e. R is maximal in SE.

Conversely, suppose $R \in \boldsymbol{R}$, and take any $t \in D'$. By (11), the set $R' = R \cup R \bullet \{t\}$ is a proper ideal of the lattice $SE' = SE \cup SE \bullet D'$. Hence by (12) that ideal is included in the sublattice $SE \cup SE \bullet \{t\}$; and in no other one of that form. Consider now any ideal I' of SE', suppose $R' \subset I'$, and take an arbitrary $z \in I'$. Then either $z \in SE$, or $z = y \vee t$, for some $y \in SE$, $t \in D'$.

Suppose $z \in SE$. Then $x \vee z \neq \lambda$, for any $x \in R$, as both are in I', and I' is proper. This means in turn that there is in R no "impedance" to z (cf. [1], proposition 5.1), and so $z \in R$!

Thus suppose $z = y \vee t$. By (12), I' is included in the sublattice $SE \cup SE \bullet \{t\}$; and so t must be the same atom of D' as before. Should z lie outside of R', its component y would have to lie outside of R (cf. Diagram 3). But y is compatible with every member of R, as otherwise I' could not be proper, containing both y and R. Hence y lies inside of R, and so $z \in R'$!

I.e., on both counts we get: $R' \in \boldsymbol{R'}$.

Note: Epistemologically, the adjunctive extension of a frame F into another F' means revealing a new aspect of reality suppressed in F, and not yet reflected in the language of F'. For $e/D'/ = \{o\}$.

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