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ATOMS IN SEMANTIC FRAMES*

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Abstract

Elaborating on Wittgenstein's ontology of facts, *semantic frames* are described axiomatically as based on the notion of an *elementary situation* being the *verifier* of a proposition. Conditions are investigated then for such frames to be *atomic*, i.e. to have lattice-theoretic counterparts of his "*Sachverhalte*".

1. Preliminaries

This paper continues some trains of thought started in [3]. Except for an expanded proof, these "Preliminaries" are a summary of Chapter 4 there.

1.1. Semantic Frames

Consider an arbitrary sextuple of the form:

$$((L, Cn), (SE, \boldsymbol{R}), (Z, Z_0)),$$

where L is a set, Cn is a closure operation on L, SE is a set again, $\boldsymbol{R}$ is a collection of sets, Z is a function from $\boldsymbol{R}$ to $P(L)$, and Z_0 is a subset of L.

Under the intended interpretation the couple (L, Cn) is an *abstract logic*, where L is a propositional language (an algebra of formulae), and Cn is its consequence operation; the couple (SE, $\boldsymbol{R}$) is an *ontology*, where the members of SE are *elementary situations*, and those of $\boldsymbol{R}$ are *realizations* ("possible worlds"); the couple (Z, Z_0) is the *semantics*, where the function Z maps realizations into sets of propositions, and Z_0 is the set of all *true* propositions of L. (The logic (L, Cn) is called "abstract" as it is not related to Z_0, i.e. to the notion of truth. It turns "concrete" as soon as we relate them by the condition: if X Z_0 then $Cn\ X$ Z_0, for any X L.)

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Under the intended interpretation the sextuple indicated will be called a *semantic frame* if the following nine axioms are satisfied (one logical, five ontological, three semantical):

(L1) (*L*, *Cn*) is classic.

Observe that by (L1) there are in (*L*, *Cn*) *complete* sets of propositions and let ***Z*** be their totality.

(R1) $\boldsymbol{R} \quad P(SE)$
(R2) $\boldsymbol{R} \neq$
(R3) $\boldsymbol{R} \neq SE$
(R4) $\cap \ \boldsymbol{R} \neq$
(R5) $R_1,\ R_2\!: (R_1 \quad R_2 \qquad R_1 = R_2)$

Elementary situations in $\boldsymbol{R}$ are *possible*; those in $\Lambda = SE - \boldsymbol{R}$ are *impossible*; those in $\Omega = \cap \boldsymbol{R}$ are *necessary*; and those in $\boldsymbol{R} - \cap \boldsymbol{R}$, if any, are *contingent*. ("If any", for the axioms admit an $SE = \Omega \quad \Lambda$, with $\boldsymbol{R} = \{\Omega\}$.)

(Z1) $Z/\boldsymbol{R}/ \quad \boldsymbol{Z}$
(Z2) $Z_0 \quad Z/\boldsymbol{R}/$
(Z3) $\alpha \quad L \quad R \quad \boldsymbol{R}: (\alpha \quad Z(R) \qquad x \quad R \quad R' \quad \boldsymbol{R}:$
$(x \quad R' \qquad \alpha \quad Z(R'))).$

1.2. Some Definitions and Results

We define the *bunch* *r*(*A*) of realizations *tied* by the set $A \quad SE$:

(1) $r(A) = \{ R \quad \boldsymbol{R} : A \cap R \neq \quad \}$

Taking "*r*(*x*)" as short for "*r*({*x*})" we have then:

(2) $r(x) = \{ R \quad \boldsymbol{R} : x \quad R \}$

Two elementary situations are said to be *incompatible* (or to *exclude* each other) if no realization comprises both, i.e.,

(3) $x \quad y$ iff $r(x) \cap r(y) = \quad .$

And the set:

(4) $A \ = \{y \quad SE\!: \quad x \quad A\!: x \quad y\}$

is the *quasi-complement* of the set *A*, for any $A \quad SE$.

The operation defined as:

(5) $V(A) = (A\ )$

is a closure on $P(SE)$, and the totality of closed sets is:

(6) $\boldsymbol{V} = \{A \quad SE: A=V(A)\}$.

The members of $\boldsymbol{V}$ will be called *V-sets*. The following holds in view of axioms (R1)-(R5):

(7) Under inclusion the collection $\boldsymbol{V}$ of all V-sets is a complete complemented lattice, with inf $\ \boldsymbol{V}_i = \cap\ \boldsymbol{V}_i$, for any $\boldsymbol{V}_i \quad \boldsymbol{V}$, and with as complementation.

For any α L, the collection of realizations

(8) $M(\alpha) = \{R \quad \boldsymbol{R}: \alpha \quad Z(R)\}$

is the logical *locus* of the proposition α, and $M/L/$ is the totality of such loci. Now

(9) $V(\alpha) = \{x \quad SE: \quad R \quad \boldsymbol{R}: (x \quad R \qquad \alpha \quad Z(R)\}$

is the set of all *verifiers* of the proposition α, and $V/L/$ is the totality of such *verifier sets*.

We have then under axiom (Z3), for any $\alpha \quad L,\ R \quad \boldsymbol{R}$:

(10) $V(\alpha) \cap R \neq$ iff $\alpha \quad Z(R)$;

and consequently:

(11) $r(V(\alpha)) = M(\alpha)$.

Under inclusion, $V(\alpha)$ is the greatest set in SE to intersect exactly all the realizations in $M(\alpha)$.

All verifier sets are V-sets. And we have:

(12) $V/\boldsymbol{L}/$ is a {0,1} – sublattice of $\boldsymbol{V}$.

Moreover, by (11) we get:

(13) $(V/\boldsymbol{L}/, \quad)$ is isomorphic to $(M/\boldsymbol{L}/, \quad)$.

Since $M/\boldsymbol{L}/$ is a field of sets we get eventually:

(14) As a sublattice of $\boldsymbol{V}$, the collection $V/\boldsymbol{L}/$ is a Boolean algebra.

Henceforth the lattice of all *V*-sets in a semantic frame will be called a *V-lattice* for short.

1.3. Entailment and Inseparability of Elementary Situations

In the universe *SE* the collection ***R*** induces a relation of *entailment*, to be written “$x \vdash y$”, and read “*x* entails *y*”:

(1) $x \vdash y$ iff $r(x)$ $r(y)$.

Clearly this is a *quasi-ordering*, i.e. reflexive, transitive, and non-symmetric. (The last is to say that there are some *x*, *y* in *SE* such that $x \vdash y$, but not conversely, e.g., any x Λ, y Ω will do here.)[1] In “$x \vdash y$” the element *x* is the “upper”, and *y* is the “lower” one. Evidently, every realization is closed downwards: for any x, y SE, and any R ***R***:

(2) x R $(x \vdash y$ y $R)$.

Furthermore, we have a relation of *inseparability* defined as:

(3) x $_r y$ iff $r(x) = r(y)$.

Clearly this is an equivalence, partitioning *SE* into blocks of inseparable elements in respect to ***R***. Both Ω and Λ are such blocks: if x Ω, then $/x/ = \Omega$, and if y Λ, then $/y/ = \Lambda$.

Defining as usual:

(4) $/x/ \leq /y/$ iff $y \vdash x$,

we see that

(5) $(SE,$ $_r, \leq, \Omega, \Lambda)$ is a bounded partial ordering, with Ω being its zero, and Λ its unit.

Observe that all realizations are unions of blocks, viz., for any x SE, R ***R***:

(6) x R iff $/x/$ R, and so $R =$ $\{/x/: x$ $R\}$.

[1] Note that we use the term *quasi-ordering* in a somewhat special sense.

1.4. The Axiom of Conjunction

The five ontological axioms adopted thus far tell us nothing about the inner structure of realizations, except that these are sets of elementary situations. Now we are to adopt some axioms pertaining to that structure.

First comes the *Axiom of Conjunction*. Setting *Fin*(*X*) as the totality of the finite subsets of *X*, including the empty one, we postulate:

(R6) $A\ \ Fin(SE)\ \ \ z\ \ SE\ \ \ R\ \ \mathbf{R}: (z\ \ R \text{ iff } A\ \ R).$

Under (R6) we have then (with "*A*/*r*" as short for "*A*/ *r*"):

(1) The partial ordering (SE/r, ≤) is a join-semilattice,

as by (R6) the join $/x/\ \ /y/ = \sup\{/x/, /y/\}$ always exists.

Clearly, any set of the form *R*/*r* is an ideal of that semilattice.

A semantic frame satisfying axiom (R6) will be called a *conjunctive frame,* and its *V*-lattice is a *conjunctive V-lattice*.

As it has been shown by Professor Wroński (cf. [3], p. 315), the following remarkable condition holds:

(*) Every conjunctive *V*-lattice is distributive.

Wroński's proof (reproduced in [3] verbatim) skips some steps which may be not quite obvious to all, so let us put them in here. The main one is the equivalence:

(**) *Under (R6):* $V_1\ \ V_2$ iff $V_1 \cap V_2\ \ \ \Lambda$, for all $V_1, V_2\ \ \mathbf{V}$.

To see this observe that for any $A\ \ SE$ its quasi-complement A is the greatest set to exclude it: $A\ \ B$ iff $B\ \ A$. Cf.([3], p. 155). So

(1) $V_1\ \ V_2$ iff $V_1\ \ V_2$.

But we also have (for proof see. [3], p. 161):

(2) *Under (R6):* $V_1 \cap V_2 = \Lambda\ \ \ V_1\ \ V_2$, for all $V_1, V_2\ \ \mathbf{V}$.

Hence we get:

(3) *Under (R6):* $V_1 \cap V_2 = \Lambda\ \ \ V_1\ \ V_2$, for all $V_1, V_2\ \ \mathbf{V}$.

Noting that for V-sets $V_1 \cap V_2 \quad \Lambda \quad V_1 \cap V_2 = \Lambda$, we see that in (**) implication holds.

On the other, hand we have, for any $A \quad SE$:

(4) $A \cap A \quad \Lambda.$

(Indeed, if $A=\ $, the inclusion holds. And if $z \quad A \cap A$, for some $z \quad SE$, then we have: $x \quad z$, for any $x \quad A$, as $z \quad A$. And also as $z \quad A$, we obtain from the foregoing: $z \quad z$, which means that $z \quad \Lambda$ (cf. [3], p. 156).) So for any $A,B \quad SE$:

(5) $B \quad A \qquad A \cap B \quad \Lambda.$

(Indeed, if $B \quad A$, then $A \cap B \quad A \cap A$; so $A \cap B \quad \Lambda$ by (4).) Hence by substitution: $V_1 \quad V_2 \qquad V_1 \cap V_2 \quad \Lambda$, which is the other implication of (**). QED.

As for Wroński's proof. He starts with the lemma:

(*a*) In a conjunctive V-lattice, for any A, B, $C \quad V$:
$A \quad B + C$ iff $A \cap B \quad C$

where + is the join, defined as $B + C = (B \quad C)$

To show it he points out that the following five formulae are mutually equivalent:

(i) $A \quad B + C$
(ii) $A \quad (B \quad C)$
(iii) $A \quad (B \ \cap C\ )$
(iv) $A \cap B \ \cap C \quad \Lambda$
(v) $A \cap B \quad C.$

Indeed, (i) is equivalent to (ii) by definition; (ii) is equivalent to (iii) by the identity $(B \quad C) \ = B \ \cap \ C$, valid for all $B,C \quad SE$; (iii) is equivalent to (iv) by (**), under the substitutions V_1/A, $V_2/B \ \cap C$; and (iv) is equivalent to (v) by (**) again, substituting this time $V_1/A \cap B$, $V_2/\ C$, and then observing that $C \quad = C$, as C is a V-set.

Secondly, he points out the inclusion:

(*b*) $(A + C) \cap A \quad B.$

Indeed, as any set of the form *X* , for an *X* *SE*, is a *V*-set, so is (*A* *B*) . Its complement in the lattice of *V*-sets is (*A* *B*) = *A* + *B*, and their meet is (*A* + *B*) ∩ (*A* *B*) = Λ.

Now taking this last equality as our starting point we can see that it is equivalent successively to each of the following formulae:

(*A* + *B*) ∩ *A* ∩ *B* = Λ (by the identity mentioned above)
(*A* + *B*) ∩ *A* (*B*) (by (**), V_1/(*A*+*B*) ∩ *A* , V_2/ *B*),

thus establishing (*b*) as *B* =*B*. The rest of the proof is easy.

1.5. The Axiom of R-Compactness

Next we adopt an axiom which in view of its form we call the *Axiom of R-Compactness*:

A *SE* ((*B* *Fin*(*A*) *R* ***R***: *B* *R*) *R'* ***R***: *A* *R'*).

Under both (R6) and (R7) we have then, for any *y* *SE*, *R* ***R***:

(1) *y* *R* *z* *R*: *y* *z*,

i.e., if an element does not belong to a realization, then there is in the latter some *impedance* to it.

Under both (R6) and (R7) we also have:

(2) For any *R* ***R*,** the set *R*/*r* is a maximal ideal of the semilattice *SE*/*r*, and all maximal ideals of the latter are of that form.

By construction the join-semilattice *SE*/*r* is *separated* by its maximal ideals, i.e., for any /*x*/≠/*y*/ there is some *R* ***R*** such that either /*x*/ *R*/*r* and /*y*/ *R*/*r*, or the other way round.

2. Atomicity

2.1. Atomic Frames

For the quasi-ordering (*SE*, ⊢) we define the notion of a *quasi-atom*, denoting their totality by *QA*. So, for any *x* *SE*:

(1) *x* *QA* iff *x* Ω and *y* Ω: (*x* ⊢ *y* *y* ⊢ *x*).

Thus quasi-atoms are elementary situations such that below of them there are only the members of Ω.

A quasi-ordering (SE, $\vdash$) will be called *quasi-atomic* if for every $y \quad SE-\Omega$ there is an $x \quad QA$ such that $y \vdash x$.

Evidently, if x is a quasi-atom of the quasi-ordering (SE, $\vdash$), then its equivalence class $/x/_r$ is an *atom* (in the usual sense) of the partial ordering (SE/r, $\leq$), i.e., we have:

(2) $\quad x \quad QA \qquad y \quad SE$: $(/y/\leq/x/ \qquad (/x/=/y/$ or $/y/=\Omega))$.

Indeed, if $x \quad QA$, then $x \vdash y \qquad (x \;\; _r \; y$ or $y \quad \Omega)$, by (1). And hence we get the consequent of (2).

Thus if the quasi-ordering (SE, $\vdash$) is quasi-atomic, then the partial ordering (SE/r, $\leq$) is atomic (in the usual lattice-theoretic sense). In that case we shall also say that the semantic frame itself is *atomic*. (The quasi-atoms are packed in blocks of inseparable ones, each of which is an atom of SE/r.)

In the following we adopt several *Postulates of Atomicity* intended to capture between them, and at this level of generality, the essence of the philosophy of Logical Atomism. However, we shall not strive here to have them mutually independent, postponing that question to a separate investigation.

Henceforth let SF be an arbitrary semantic frame as characterized above. We adopt:

(A1) $\quad SF$ is atomic,

as our first postulate; i.e., every elementary situation which is not necessary entails some quasi-atomic ones.

Postulate (A1) is not to preclude the possibility of *atomless* frames. On the contrary, we even regard them as more realistic. The point of adopting (A1) is merely the consideration that atomic frames are easier to handle mathematically. Thus, to begin with, they may do as a first approximation. And we set:

(3) $\quad$ For any $A \quad SE$ let $At(A)$ be all the quasi-atoms which A entails; i.e., $At(A) = \{z \quad QA: \quad x \quad A: \; x \vdash z\}$.

Now, as our second postulate, we assume that every realization is uniquely determined by its quasi-atoms; i.e., (A2) R_1,R_2: $(At(R_1) = At(R_2) \quad R_1=R_2)$.

A semantic frame satisfying both (A1) and (A2) will be called *atomistic*. Only atomistic frames are to be considered in the following.

2.2. Logical Dimensions

Consider any set $A \quad SE$. We shall call that set *exclusive* if any of its distinct elements exclude each other; i.e., if we have:

$$x,y \quad A: (x \neq y \qquad x \quad y).$$

Clearly, if $A \quad B$, and B is exclusive, so is A.

We shall call a set A *transverse* (over the logical space $\boldsymbol{R}$) if it intersects all realizations; i.e., if we have:

$$R \quad \boldsymbol{R}: A \cap R \neq \quad ; \qquad \text{or equivalently:} \qquad \{r(x) : x \quad A\} = \boldsymbol{R}.$$

Clearly, if $A \quad B$, and A is transverse, so is B. Trivially, the set Λ is exclusive, and so is any singleton $\{x\} \quad SE$. The set SE is transverse, and so are Ω and QA, the latter due to (A1).

An atomistic frame is *dimensionally determinate* if there is a partition $\boldsymbol{D}$ of QA such that each of its blocks is both exclusive and transverse. The blocks $D \quad \boldsymbol{D}$ are then the "dimensions" of the logical space $\boldsymbol{R}$, or *logical dimensions* for short. Our third postulate is:

(A3) *SF* is dimensionally determinate.

Observe that for a set A which is both exclusive and transverse, any intersection with a realization must always be a singleton (never plural by the former, and never empty by the latter). So we have:

(1) $R \quad \boldsymbol{R} \quad D \quad \boldsymbol{D} \quad x \quad QA: D \cap R = \{x\}$.

Observe that in view of (1) logical dimensions are sets of quasi-atoms which are both maximally exclusive and minimally transverse. Indeed, suppose D, as an exclusive set of quasi-atoms, is not maximal. There should be then an $y \quad QA$ such that $x \quad y$ for all $x \quad D$. However, being a quasi-atom, y must belong to some realization R, and this has to be such that $D \cap R = \quad$, contradicting transversity. On the other hand, suppose D, as an transverse set of quasi-atoms, is not minimal. There should be then an

$x \quad D$ such that $D-\{x\}$ is still transverse. However, this would mean – as x is a quasi-atom, $r(x)$ is never empty – that for every $R \quad r(x)$ there is an $y \quad D-\{x\}$ such that $R \quad r(y)$. Taking any such y we see that for some $R \quad \boldsymbol{R}$ we would have $x, y \quad R$, contradicting exclusiveness.

2.3. Wittgenstein

Our “quasi-atoms” are to be formal counterparts of Wittgenstein’s “states of affairs” (*Sachverhalte*). These in turn were to be the reference of his “elementary propositions” (*Elementarsätze*). In the *Tractatus* it is stipulated that they be mutually independent (2.061), and even totally so (4.27); in particular no state of affairs is to entail, or to exclude any other (2.062).

Later, however, Wittgenstein got second thoughts here, and around 1929/30 he started moving in the very direction we consider here: towards segregating states of affairs into sets of mutually exclusive ones. But he never had had much patience with elaborating formal details, and in view of mounting difficulties – which he certainly must have encountered – the idea was dropped. Nevertheless traces of it may be found in his writings of that period, and we are going to present a survey of them now. The relevant passages will be quoted at some length, as to our knowledge they have hardly been discussed up to now, and are not even accessible in English. They are contained mostly in Wittgenstein’s “*Philosophische Bemerkungen*”, written between February 1929 and July 1930, and published in 1964; and in his *Schriften 3*, published in 1967 and containing his talks with Friedrich Waismann in the period mentioned. (Translations from German will be ours.)

The gist of Wittgenstein’s change of mind was his idea of a “system of propositions” (*Satzsystem*). This may be defined as a set of elementary propositions such that of necessity one and only one of them is true. Thus the reference of such a “system” is a “logical dimension” of reality in our sense of the term. He had introduced the idea in the talk with Waismann of 25.12.1929 (cf. [2], p. 63-64):

> “Once I had written: “propositions are like rulers applied to reality”. Now I would rather say: systems of propositions are like a ruler applied to reality. /…/ Statements describing the length of an object form a system – a system of propositions. The entire system is compared with reality, not single propositions. If I say that, e.g., a particular point in my field of vision is b l u e, I know not only this, but also that it is not green, not red, not yellow, etc. I have applied the entire range o f

> colours all at once. /…/ When a system of propositions is applied to reality, this is already to say that just one state of affairs can obtain, never several of them.
>
> I used to think elementary propositions should be independent: from a state of affairs obtaining one cannot infer that another does not. But if my present ideas about systems of propositions are right, then it is even the rule that from one state of affairs obtaining we can infer that no others described by the system do."

If we do not ask for details, the general idea seems clear enough. He elaborated a bit on it ten days later, in the talk of 5.1.1930 ([2], p. 89-91), using even the term "dimension" there:

> "Statements of length are always part of a system. For to understand that something is 3 m long implies understanding also what it means to say that it is 5 m long. The statement is already part of a system of possible lengths. Similarly things have a space of colours around them, of hardnesses, etc. However, writing this /cf. *Tractatus* 2.0131/ I did not realize that positions in that space are like marks on a ruler, and that – as with the ruler – we always apply to reality an entire system of propositions. The general question is: does proposition "φ*a*" presuppose other propositions of this kind, e.g. "φ*b*"?
> (Whatever there is, might be different. Reversal: there is only what might also be different.)
>
> The question comes down to whether the sign "*a*" is an indispensable one. Should there be only the proposition "φ*a*", and no "φ*b*", then mentioning "*a*" would be superfluous. Writing just "φ" would do. The proposition would not be compound. However, propositions are essentially images, hence compound. Thus if "φ*a*" is to be a proposition, there must also be a proposition "φ*b*", i.e. the arguments of "φ ( )" form a system. What is not known here, is the size of the range of those arguments. There might be, e.g., just two of them. (Telephone dial: free, busy. Here we know: just these two values are there, and they are what reflects reality. Intermediate positions mean nothing. No transitions.)
>
> But does "φ*a*" also presuppose "ψ*a*"? Certainly. For by the same reasoning: should there be to "*a*" just the one function

> "φ", it would be superfluous and might be dropped. The propositional sign would then be simple, not compound. No mapping then. Dispensable signs mean nothing."

Lots of questions arise. There is, to begin with, an ambiguity. Let the set $\{a_1,\ldots,a_i,\ldots\}$ be the range of the function "φ ( )", i.e., the totality of its meaningful substitutions. What is the "system" then: the set $\{a_1,\ldots,a_i,\ldots\}$ itself, as it is explicitly stated in that passage ("the arguments of "φ ( )" form a system"); or rather – as the foregoing passage would indicate – the set of propositions {"φa_1",…," φa_i",…} obtained by substituting in the function mentioned the members of the former; i.e. not the arguments, but the values of the function "φ ( )"? In the first case the system is formed by some objects, in the other by some states of affairs. Or does that make no difference?

Secondly, how are the various "systems of propositions" related to each other. Presumably they are disjoint as sets. Thus, if every elementary proposition belongs to some system, there should be a partition of them into blocks, each block a system. But is it unique? Suppose "*p*" and "*q*" are two elementary propositions, "*p*" belonging to the system S_1, and "*q*" belonging to the system S_2. And suppose they are strictly equivalent: whenever "*p*" holds, so does "*q*", and *vice versa*. Then swapping them would yield two systems again: $S_1' = (S_1 - \{p\})\quad \{q\}$, and $S_2' = (S_2 - \{q\})\quad \{p\}$. Partitioning would be preserved, but in a different way. How are we supposed to deal with such cases?

Presumably the systems should be mutually independent (or "orthogonal", as we have put it in [3], p. 27). But totally so, or just in pairs? This needs clarification. And what about combining the systems into products, e.g. $S_1 \cdot S_2$, with their members being conjunctions of the form "*p q*"? Is every product a system again? Then what about infinite collections of systems? Should there be some kind of topological compactness in operation for them?

At any rate not elementary propositions only may be partitioned into systems; other kinds of propositions are eligible as well. Moreover, we have considered conjunctions only, but what about disjunctions? Obviously, all this needs expansion into a full-blown theory, but apparently Wittgenstein had no patience with it, or maybe he even did not feel up to it any more. Instead he plunged into the pointless vagaries of his "later" philosophy, starting with the "Blue Book" in 1933. The quality of his thinking in the years 1929/30 may not have reached the summits of the *Tractatus*, but still it is far superior to anything he had produced later.

As we have seen, in the passage just quoted Wittgenstein says:

> "If 'φ*a*' is to be a proposition, there must also be a proposition 'φ*b*', i.e. the arguments of 'φ ( )' form a system. What is not known here, is the size of the range of those arguments."

This is to say that in order to grasp the sense of the proposition "φ*a*" we have to know at least one element ***b*** such that the proposition "φ*b*" would be incompatible with "φ*a*". Only this makes the negation of "φ*a*" intelligible to us. If we know just one such ***b***, then to us the negation "¬ϕ*a*" is simply equivalent to "φ*b*"; and if we know more of them – e.g. ***b*** and ***c*** – then to us the negation "¬φ*a*" is equivalent to the disjunction "φ*b* or φ*c*", opposed to "φ*a*". (As the Schoolmen used to say: *eadem est scientia oppositorum*.)

Thus to grasp the sense of the proposition "φ*a*" we need not know all the meaningful substitutions in "φ ( )"; i.e. not all the objects whose names, if there were any, would turn the propositional function "φ ( )" into a meaningful proposition. To grasp, e.g., the sense of "this is red", we have to be acquainted with at least one different hue, but we need not by any means to be acquainted with all of them! (There may be exotic colours we have never seen, nor even imagined.)

In the talk of January 5th, 1930 – in which also Moritz Schlick participated – Wittgenstein raised that question too ([2], p. 88-89):

> "Let me pick up Professor Schlick's question again: how would it be if red were the only colour we know. The following is to be said here: if all we see were red, and we could describe it, then we should also be able to form the proposition that it is not red. And this presupposes already the possibility of other colours. Otherwise red would be something we cannot describe – for we have then no proposition, so there is nothing to deny. In a world where red plays quasi the same part as time does in ours, there could be no statements of the form "everything is red", or "whatever we see, is red".
> Thus: if we deal with a state of affairs, then it may be described; and the colour red presupposes then a system of colours. Otherwise red means something quite different, and it makes no sense to call it a colour.

> (“The world is red”: if this can be said in a proposition, then it may also be denied; and the proposition lies in a space. If, however, no proposition can describe that, then we are unable even to ask whether a system of colours is presupposed by red.)”

Thus Wittgenstein’s “systems of propositions”, or at least some of them, would be relative to the scope of experience of the speaker, and they might differ from one speaker to another. If I know what blue is, I have to know at least one hue opposite to it, say red; but with somebody else it may be a different one, say green, or yellow. Negation needs a positive paradigm to give it meaning.

But whatever one’s system of propositions associated with the proposition “φ*a*”, it always has to be complete, i.e. the references of its members have to form a “logical dimension” in our sense of the term. And this points already to the feasibility of extending one’s semantic frame – in the case discussed, one’s semantic frame of colours – by switching to a system of propositions with more hues.

Observe, however, the following. Let S_1 be the original system, and S_2 the expanded one. They both have to be complete, and so S_1 cannot be *part* of S_2; it must be its homomorphic image, its diminutive effigy. Thus it has to contain some dummy colour, representing in it whatever other colours might be, and giving it completeness thereby. The transition from S_1 to S_2 has to be not as shown in Figure 1, but as in Figure 2.

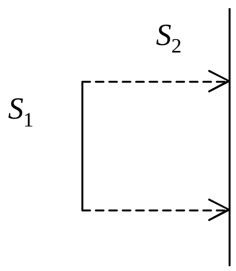

Figure 1

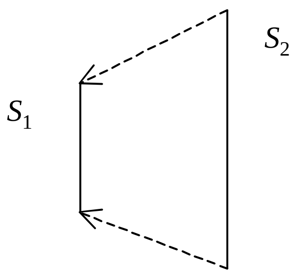

Figure 2

The same topics are depicted in Chapter VIII of [1]. There we read:

> If different degrees exclude each other, then the presence of one entails the absence of another. Thus two elementary propositions may contradict each other.
>
> How possibly can *f*(*a*) contradict *f*(*b*), as seems to be the case? E.g., when I say "now it is red here", and "now it is green here"?
>
> When *f*(*r*) contradicts *f*(*g*), it is because the *r* and *g* fill in the *f* completely, and don't go in there both. This, however, does not show up in our signs. But it should, if the symbol were considered, not the sign. For as the former comprises the form of objects, the impossibility of "$f(r) \cdot f(g)$" must show up there, in that very form. (pp. 106-107.)
>
> What I apply like a ruler to reality, is not a proposition, but a s y s t e m of propositions.
>
> The concept of an "elementary proposition" looses now most of its former importance.
>
> The concept of the independent co-ordinates of description: propositions connected e.g. by "and" are not mutually independent; they form o n e image, and may be checked as to their compatibility or incompatibility.
>
> On my former view of elementary propositions there was no determining the value of a co-ordinate, though my remark about coloured bodies being in a space of colours, etc., should have led me right there.
>
> What we come now to realize is that we deal here with rulers, not with their individual marks.
>
> Thus every proposition would be like adjusting a set of rulers. /…/ Consider, e.g., the signals on board of a ship: "stop", "full ahead", etc.
>
> But they need not be rulers. A dial with two signals on it would hardly be called a ruler. (pp. 110-112.)

Wittgenstein seems not to have realized that his idea of "a system of propositions" is not so radically different from what he had thought before. Any couple of propositions: one elementary, the other its negation, constitutes a system of propositions in his sense, namely a b i n a r y one. (Cf. [3], pp. 29-32.) What corresponds in reality to the non-existence of a state of affairs is not some sort of a metaphysical g a p, but another state of affairs filling in that gap in place of the other. (Like the system "wet - dry", where "dry" is equivalent to "not wet".) What was wrong with the *Tractatus* view was only the assumption that all systems of elementary propositions are of that binary kind. But they are the exception, not the rule.

Wittgensteinian “systems of propositions” are represented in language by certain propositional functions. These arise by turning certain constituents of elementary propositions into variables. But w h a t constituents? If in “Jack loves Jill” we turn the constituent “Jack” into a variable, we get the function “*X* loves Jill”. But the substitution “Tom loves Jill” does not belong to the same system of propositions as the former one, for they are perfectly compatible. On the other hand, if in “Jack’s desk is 172 cm long” we turn “172” into a variable, and get “Jack’s desk is ***n*** cm long”, then the latter determines a system of propositions, its members being substitutions of all the integers taken from – say – the interval “100” to “300”.

What is the difference between the two cases? Certainly not the circumstance that one constituent is a numeral and the other a personal name. (For “*X* is Jill’s father” determines a system of propositions all right, consisting of the substitutions of all male names.) There is surely food for thought here.

2.4. Finite Atomicity and Q-spaces

Let us define a relation $K(z,A)$, with $K \quad SE \times P(SE)$, and to be read as “the element *z* is *conjunctively associated* with the set *A*”, i.e.,

(1) for any $A \neq \quad : K(z,A)$ iff $\quad R \quad \boldsymbol{R}: (z \quad R$ iff $A \quad R)$.

Now the Axiom of Conjunction (R6) may be put down succinctly as follows:

(2) $A \quad Fin(SE) \quad z \quad SE: K(z,A)$.

As a next step we introduce the Postulate of Finite Atomicity for the semantic frames under consideration:

(A4) $z \quad SE \quad A \quad Fin(QA): K(z,A)$,

i.e., every elementary situation is conjunctively associated with some f i n i t e set of quasi-atoms.

What we are up to now is to secure two things. Firstly, that there always be a m i n i m a l set of quasi-atoms to satisfy (A4), i.e. such that for any set *A’* smaller than *A* the relation indicated would not hold. Secondly, to achieve this we need a closer relationship (than that of proposition (2.1)) between the notions of a realization, i.e. the collection ***R***, and that of a logical dimension, i.e. the collection ***D***.

A semantic frame satisfying (A4) we call *finitely atomistic*. The corresponding partial ordering $(SE/r, \leq)$ is then finitely atomistic in the sense defined in [3], p. 307 sqq.

As an intermediary link between $\boldsymbol{R}$ and $\boldsymbol{D}$ we introduce the notion of a $\boldsymbol{Q}$-*space*, denoting their totality by $\boldsymbol{Q}$ (cf. [3], p. 37 sqq). Any Q-space $Q \quad \boldsymbol{Q}$ is a set of elementary situations both exclusive and transverse, so that the logical dimensions turn out to be a special case: $\boldsymbol{D} \quad \boldsymbol{Q}$. Moreover, we have:

(3) $\quad\quad \boldsymbol{Q} = \quad \boldsymbol{R}.$

We want, and this is the tricky point, to have $\boldsymbol{Q}$ and $\boldsymbol{D}$ related in such a way that to each $Q_i \quad \boldsymbol{Q}$ there corresponds exactly one finite set of logical dimensions, and vice versa. I.e., we postulate a one-one correspondence:

(4) $\quad\quad \boldsymbol{Q} \leftrightarrow Fin\,(\boldsymbol{D}).$

To each elementary situation $x \quad SE - \Lambda$ there corresponds then via (4) a *dimensional index* $\boldsymbol{D}_i = D(x)$, where $D(x)$ is a collection of dimensions, enabling us to define a partial ordering on the elementary situations themselves (and not as before on their equivalence classes only). For any possible $x, y \quad SE$ we set:

(5) $\quad\quad x \leq y$ iff $(y \vdash x$ and $D(x) \quad D(y)).$

Evidently, this is both reflexive and transitive. To see that it is also antisymmetric, assume that $r(x) = r(y)$ and $D(x) = D(y)$, and suppose that $x \neq y$. By the second assumption x, y belong to the same Q-space, hence being different they exclude each other: $x \quad R \quad\quad y \quad R$, for all $R \quad \boldsymbol{R}$. By the first one, however, we get the opposite: $x \quad R \quad\quad y \quad R$, for all $R \quad \boldsymbol{R}$. Consequently $x \quad R$, for all $R \quad \boldsymbol{R}$, contradicting the initial assumptions of x being a possible situation.

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