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CONCEPTIONS OF LOGICAL IMPLICATION*

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Abstract

This is a survey paper of approaches to the concept of logical implication. Roughly stated the main motivation of these approaches is to provide a necessary and sufficient condition for a set of propositions to logically imply a single proposition. In regard to their affinities these approaches are grouped into two: the *transformational* conception and the *informational* conception. Some approaches in each conception are philosophical and some are mathematical in character, their common assumption being that they reflect a previous intuition – in some cases – already at work in classical mathematics. The first part of the paper is devoted to approaches within the transformational conception of logical implication emphasizing the contributions of Bolzano, Russell, Tarski, Quine and current model – theory. The second part of the paper discusses approaches within the informational conception of logical implication. Carnap and Bar Hillel's extrinsic approach and Corcoran's intrinsic approach developed in his information-theoretic logic are respectively compared and some of their philosophical underpinnings are brought to light. The paper concludes with some final remarks concerning the ontic question of the nature of logical implication and the epistemic question of the human access to this relation in both conceptions of logical implication. The article also attempts to connect the views of these representative thinkers with the current philosophical debate on the nature of logical consequence and the role of set-theoretic model theory in philosophy of logic.

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1. Notational and terminological preliminaries

Single quotes are used to indicate that the discourse is about expressions, whether single words or sentences. Double quotes are used to indicate that the discourse is about meanings, whether concepts or propositions. Thus, the sentence ‘two is oblong’ expresses the proposition “two is oblong”. Occasionally, *italics* are used for purposes of emphasis.

In the discourse of this paper an *argument* is a two-part system composed of a set of propositions **P** (the premise-set) and a single proposition **c** (the conclusion). Arguments are valid or invalid. In order for an argument **P, c** to be valid it is necessary and sufficient that **P** logically implies **c**. Notice that the present characterization of arguments is purely ontic. It does not involved any subject or agent deducing **c** from **P** or producing a chain of reasoning **R** going from **P** to **c**. The word ‘argumentation’ is reserved for another sense of ‘argument’ that sometimes is found in the literature. An argumentation involves a chain of reasoning or chain of inferences intended to show that a certain conclusion is indeed a consequence of a given set of premises. Argumentations are conclusive (cogent) or inconclusive (fallacious).

It is customary to distinguish between sentences, or string of characters, arranged according to rules, from the propositions or thoughts expressed by those sentences. In a similar vein a parallel distinction between *argument-texts* and arguments is adopted. In a strict sense it is arguments and not the syntactic argument-texts that express them under various interpretations that are properly said to be valid or invalid. However, accurate description of some viewpoints relevant for the goal of this paper requires elliptical predication of validity or invalidity of argument-texts.

The term ‘implication’ can be used as a relational expression between appropriate entities, for example when it is said that the relation of implication is independent of the truth-values of the propositions involved. ‘Implication’ can also be used as a common noun, designating a proposition that is implied by a set of propositions, the implicants. For example “two is even” is an implication of the implicant proposition “two is oblong and two is even”. In the discourse of this paper ‘implication’ is used for the most part as a relational verb. The expression ‘logical consequence’ is used to denote the converse relation of logical implication. Thus, that a given proposition is a logical consequence of a set of propositions and that a given proposition is logically implied by a set of propositions, amounts to the same thing.

It should be clear by now that the relation of implication is prior to the relation of deducibility. Deducing a conclusion **c** from a premise-set **P** through a chain of reasoning **R** is rather the epistemic criterion to determine the existence of the relation of logical implication between **P** and **c**. A deduction establishes for the agent that the argument **P, c** contained is valid. Deducing requires an agent. An agent **X** establishes the validity of the argument **P, c** by a chain of reasoning **R** going from **P** to **c**. If **R** does not contain mistakes **X** has found a deduction. Thus, 'deduction' is a *successful* term in the present discussion. Thus, every deduction is an argumentation (a cogent one), but not every argumentation is a deduction.

2. Motivation: the importance of the issue of logical implication

Logical implication has a foundational role in most aspects of rational life as it can easily be seen in reading-comprehension of a given text or at the time of settling a given hypothesis. In effect, understanding a text amounts to answering the question for the class of consequences that follow from the given text. In the course of an investigation, determining the truth of a given unknown proposition amounts to deducing it from premises already known to be true; i.e., amounts to establish the existence of the relation of logical consequence between the hypothesis and the premise-set. Likewise, in order to determining the falsehood of a given unknown proposition it is sufficient to deduce from it (perhaps in addition with other premises already known to be true) a proposition already known to be false. It amounts to establishing the existence of the relation of logical consequence between a false proposition and the given hypothesis. Thus, both, the deductive and the hypothetic-deductive method of settling a hypothesis rest on logical implication.

Each approach within the conceptions of logical implication to be examined postulates a candidate for the necessary and sufficient condition for a proposition to be a logical consequence of a set of propositions. Of course, from the philosophical angle, this is just part of the issue. To have the right extension of a concept is one thing, quite another is to have a good explanation or an *analytically* adequate definition of the given concept. From our present point of view, many of these approaches seem to have postulated a necessary but not a sufficient condition for an argument to be valid, in this sense not having obtained the material adequacy of the concept. Nor the analytical adequacy of the purported definition seems to have been properly accounted if the philosophical demand for an account of the *modality* involved in the concept of logical implication should be properly considered. These two desiderata in turn called at least, for the clarification of the issue of logical form and the more philosophically loaded issue of the kind of necessity involved in most of the intuitive accounts of logical implication.

3. Approaches within the Transformational Conception

A common feature of these approaches is that they *reduce* the relation of logical implication to one form or another of *generality*. These forms of generality are obtained when considering *transformations* or *substitutions* defined upon certain entities, let them be, expressions, concepts, expressed concepts, sequences, or pure sets, which are generally called non-logical notions. This principle of generality (**PG**) can be formulated as follows:

> **PG.** In order for an argument **P, c** to be valid it is necessary and sufficient for *every* argument **P***, **c*** obtained by appropriate and uniform transformations upon the non-logical parts of the given argument to be materially valid.

Approaches based on **PG** are reductionist in the sense that the modal component of the relation of logical implication is *explained* in terms of the material validity of *every* argument obtained by a suitable transformation. All these accounts – sometimes also called *quantificational accounts* of logical implication in regard to the presence of the universal quantifier in **PG** – are supported by the principle of form (**PF**):

> **PF.** Two arguments in the same logical form are both valid or both invalid.

It follows that every argument in the same form as a valid argument is valid and every argument in the same form as an invalid argument is invalid. **PF** sustains the so proclaimed principle of economy of thought: once a deduction is discovered in a given domain of investigation, every other interpretation in which the premises are true it also renders the conclusion true. An argument whose conclusion has been deduced from its premise-set is – so to speak – *exportable* in virtue of form. Likewise for the negative part; a suspicious argumentation can be proved to be fallacious by showing that the premise-set and its conclusion is in the same form as an already known to be invalid argument. In the strict sense there are two cases here. An argument is invalid *by fact* when it has all true premises and false conclusion. Establishing invalidity in this case simply requires knowledge by the agent of the actual truth-values of premises and conclusion. Establishing invalidity of an argument not belonging to the category of all true premises and false conclusion requires **PF**. This is accomplished by an invalid argument in the same logical form as the given argument whose premises are all known to be true and whose conclusion is known to be false. Notice that to be in the same form as an invalid argument is a necessary but not a sufficient condition for *establishing* invalidity.

The account of the modality involved in logical implication seems to be at the core of the second desiderata of a good account of logical implication. In effect, in most common approaches to logical implication we have that a proposition **c** is logically implied by a set of propositions **P** if and only if it is *logically necessary* for **c** to be true were all the premises in **P** true. Likewise, a proposition **c** is logically implied by a set of propositions **P** if and only if it is *logically impossible* for the propositions in **P** to be all true with **c** false. In this sense, the necessity conception provides the grounds for direct deduction and the impossibility conception provides the grounds for indirect deduction. These modalities seem to suggest a stronger standpoint about logical implication than the one based on mere generality. If possible worlds furnish an account of these modalities, we have that in order for **P** to logically imply **c** it is necessary and sufficient for the conclusion **c** to be true in every possible world in which the premises in **P** are all true. Likewise, in order for **P** to logically imply **c** it is necessary and sufficient for there to be no possible world in which all premises are true and the conclusion false. However, it has been pointed out that possible worlds are philosophically problematic. There are fundamental questions, yet unanswered, such as their ontological nature, existence and identity of the objects in their domains, and so forth. In other words, we seem to be providing an alleged account of the modality involved in logical consequence by means of entities of which we seem to lack proper understanding.

Given this introductory background, the views of Bolzano, Russell, Tarski in the thirties, Quine, and the contemporary model-theoretic account shall be surveyed. This overview shall shed some light on the present status of the debate within the transformational conception on the nature of logical implication and shall contribute to assess the results of this solid tradition of thought.

3.1. Bolzano: the relation of general implication

In Bolzano (1972, section 155) the relation of general implication is defined as follows: a set **C** of propositions follows from a set P of propositions *with respect to* a set I of ideas (that could contain logical or non-logical ideas according to what is suggested in section 154) if

1. The union of **P** with **C** is compatible with respect to the ideas in **I**.
2. Every substitution of the ideas in **I** that transforms **P** in a set of true propositions it also transforms **C** in a set of true propositions.

Notice in the first place that this characterization is based on the compatibility of **P** and **C**. *A fortiori*, it follows that Bolzano restricted his attention to arguments with compatible premise-sets, undergenerating thereby the class of valid arguments sanctioned by his proposal. However, it does not seem to be evidence to the effect that Bolzano excluded from consideration valid arguments with tautological conclusions.

In section 223, page 289, Bolzano suggests that in order for his definition of general implication to be coextensive to the relation of logical implication, **C** has to follow from **P** with respect to *all* the non-logical ideas in **I**:

> *Hence, what can be expected in this place is only a description of those modes of deduction whose correctness can be shown from logical concepts alone, or, what comes to the same thing, which can be expressed in the forms of truths, in which nothing is mentioned except concepts, propositions, and other logical objects.*

Thus, Bolzano took logical implication to be a species of his general implication relation. This approach clearly leads Bolzano to precede further developments in the field, particularly those established by Tarski in the thirties.

3.2. Russell: the relation of formal implication.

Russell discusses his account in a language of types within the logical framework of *Principia Mathematica*. The idea of Russell (1903/37: 4 and 14) is that the conditional sentence

‘(1 + 1 = 2) formally implies (Even(1 + 1) → Even2)’ means

$\forall x\ \forall y\ \forall P\ ((x + x) = y \to (P(x + x) \to Py))$

Similarly, the conditional sentence

‘(Even2 & ¬ Even3) formally implies 2 ≠ 3’ means

$\forall P\ \forall x\ \forall y\ ((Px\ \&\ \neg Py) \to x \neq y)$

It has been pointed out that Russell was unaware in this context of the use-mention distinction. Where he writes ‘formally implies’ the subject and the object of the relation should be appropriately set between quotes. Likewise Russell uses the word ‘proposition’ in this context to designate a construction made of words (See Russell 1919/93: 155) in strict analogy with the present use of the word ‘sentence’.

In his discussion Russell clearly presupposes a distinction between logical and extra-logical terms. However he never specifies what extra-logical terms, whether some or all, should be varied. Allowing for the best option, namely when all the extra-logical terms are subject to variation, the examples above show the Russellian formal implication reduced to a generalized material conditional of the object language. In other words, Russell would transform an argument-text into a universal closure of a material conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion, obtained by means of suitable substitutions of variables for the extra-logical or content terms. This procedure makes Russell to disregard valid arguments with infinitely many premises undergenerating thereby the class of valid arguments sanctioned by his proposal.

It is also worth noticing that in the fixed universe framework of *Principia Mathematica*, a given universal sentence, not only logically implies each of the singular instances, but also logically implies each of its numerical existentials. For example, ‘every number is positive’ logically implies ‘At least two numbers are positive’, ‘At least three numbers are positive’, etc. Of course all these are invalid from the present standpoint of model theory in which changes in the universe of discourse are contemplated. In this sense, Russell’s definition also overgenerates the class of valid arguments.

3.3. Tarski: the relation of logical consequence

In his famous 1936 article Tarski considers argument-texts in an interpreted language in the framework of *Principia Mathematica*. On the basis on the – then – merely assumed distinction between logical and extra-logical terms, Tarski indicates that for each argument-text there exists an argument-text function obtained by means of a one to one suitable substitution of variables for content or extra-logical terms in the given argument-text. Sequences provide with the values for the variables. Following Tarski terminology in his 1933 truth paper, a sequence is a function whose domain is the class of positive integers and whose values are in the entire hierarchy of types based on *the* universe of individuals whose existence is *guaranteed* by the axiom of infinity. Variables in the language considered have been enumerated in an order, and given any sequence, the k-th object in the sequence is suitable assigned as the value for the k-variable. A sequence that satisfies a set of sentential functions obtained in the previous way from a set of sentences is a *model* for that given set of sentences. Using these notions, Tarski (1936: 417) provides his famous *no-countermodels* definition of logical consequence:

> *The sentence X follows logically from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*

This definition is superior to the ones given by Bolzano and Russell regarding the level of generality obtained by incorporating extra-linguistic notions such as sequence, satisfaction and model. Likewise, Tarski was the first to make explicit in a precise manner that all extra-logical terms are to be substituted. Finally, it should be added that Tarski considered that in certain fields like mathematics, logical consequence relates conclusions with arbitrary sets of premises (perhaps infinite sets), as it is exemplified by his studies of the omega properties in his 1933b paper.

Regarded from our present standpoint however, Tarski's definition looks unsatisfactory. As it was pointed out already in Corcoran (1972), Etchemendy (1990) and Sagüillo (1997), Tarski's proposal only considered sequences generated from the given universe of the interpretation of the language. Neither he considers restricting the range of the individual variables to a subset (perhaps finite) of the universe. On the other hand, it is fair to indicate that Tarski during the thirties was perfectly aware of the notion of truth relative to different universes or domains. In effect, his 1933: 199-201 contains several passages in which he refers to the Hilbert School of logic addressing the present issue.

3.4. Quine: first-order logical truth and consequence

According to Quine, logic is first-order. For him, the underlying logic of all rational thought is standard first-order logic without identity, without individual constants and without function constants. Hence, this is a logic that merits being called conservative. His conception of the logical properties gives priority to logical truth. Roughly, Quine holds that in order for a sentence to be logically true it is necessary and sufficient for it to be true and to remain true under any uniform *lexical substitution* of its content-terms. Derivatively, in order for a sentence **c** to be logically implied by a set of sentences **P** it is necessary and sufficient for there to be no single uniform lexical substitution of content-terms that makes every member of **P** true and **c** false.

It is important not to overlook the fact that Quine takes logical truth to be prior to logical implication. In any finite universe of sentences (closed under conjunctions and conditionals), logical implication can be defined on the basis of logical truth, and conversely. The choice adopted been of no technical or philosophical significance: **P** logically implies **c** if and only if the conditional whose antecedent is the conjunction of the sentences in **P** and whose consequent is **c**, is logically true. However, the issue has import when considering infinite universes of sentences. Clearly, not every first-order argument-text admits of a suitable one-sentence translation in a standard language. Specifically, no argument-text with an infinite premise-set allows such a single sentence translation.

In several places – Quine (1936, 1940/94 and 1954/63) – Quine defines logical truth by means of three preliminary notions: *vacuous occurrence* of an expression, *vacuous variants* of a sentence, and *essential occurrence* of an expression. A vacuous occurrence of an expression in a given sentence is that for which every suitable substitution leaves the truth-value of the sentence unchanged. A vacuous variant of a sentence **S** is a sentence **S*** obtained by means of a uniform suitable lexical substitutions of occurrences of vacuous expressions in **S**. Every such vacuous variant **S*** has the same *skeleton* and truth-value than **S** has. Finally, an expression occurs essentially in a sentence **S** if it occurs in every vacuous variant **S*** of the given sentence **S**; that is to say, if it is part of the aforementioned skeleton.

Based on these notions Quine (1954/63: 357) defines logical truth, relatively to a previously specified class of logical constants in a first-order interpreted language:

> *The logical truths, then, are those true sentences which involved only logical words essentially. What this means is that any other words, though they may also occur in a logical truth (as witness 'Brutus', 'kill', and 'Caesar' in 'Brutus killed or did not killed Caesar'), can be varied at will without engendering falsity.*

Quine´s substitutional account involves three fundamental features that should not be overlooked:

1. Fixed-universe: the purported definition only considers the universe of the given interpretation of the language. In this sense, his conception can be called *local*. No other universe, neither an expansion nor a restriction of the one that is given in the intended interpretation is contemplated. Again, if regarded from our current model-theoretic standpoint this peculiarity of Quine's definition may look

unsatisfactory as Tarski's did, since logically true sentences are independent of the particular universe that the language happens to be interpreted in. In a word, this feature jeopardizes the topic neutrality of logical truth and logical implication.

2. Fixed-content: The interpretation of the language is kept fixed. No changes in the extensions attached to the non-logical terms are allowed in the present characterization. Quine thinks that logical truth is relative to a given interpreted language. In his *intra-linguistic view*, changing the language changes the concept of logical truth involved. In this sense, his concept is *immanent* rather than transcendent (See Quine 1970/86: 19-29). Thus, according to Quine, given two languages, we have two concepts of logical truth having in common that each of their interpretations are kept fixed. Perhaps this seems anachronistic to say the least.

Tarski in his seminal 1936 paper on consequence cogently argued that his semantic or *extra-linguistic* conception was superior to the syntactic or intra-linguistic conception. In effect, it is at least conceptually possible for there to be a sentence in a given language that does not have a *countervariant*, or an argument-text that does not have a *counterargument-text* due to the limited means of expression in that language. Hence, we may contrast the Tarskian no *countermodels* viewpoint of validity – involving semantic devices such as, sequences, satisfaction and model – with the Quinean no *expressible* countermodels viewpoint – involving lexical substitutions –, the first being superior to the second in the level of generality obtained. See Corcoran (1983: xi-xii).

It has been emphasized so far that the concept of logical truth and consequence developed by Quine differs from that of our current model-theoretic semantics. It is tempting to explore whether – always within Quine's framework of first-order logic – the present substitutional account can come to terms with the model-theoretic account. Indeed, in his 1970/86 book, Quine claims that both accounts are coextensive as long as two conditions are satisfied:

1. The first-order language considered must contain enough means of expression for elementary arithmetic.
2. The identity sign must belong to the non-logical vocabulary.

The first condition implies that the universe of the interpretation of the language is infinite. More importantly, the strength required for the language is also related to the – often invoked by Quine – fundamental results of Löwenheim (L) and of Hilbert-Bernays (H-B) for first-order logic. The L-theorem establishes that any set of first-order sentences that comes out true under a given interpretation it also has a true interpretation in the domain of the natural numbers. On the other hand, Quine qualifies the H-B theorem as an improvement on the L-theorem. It shows that for any given consistent set of sentences, its true interpretation in the universe of natural numbers can be expressed in the language of arithmetic. Both theorems, Quine maintains, allow us to bypass interpretations and talk directly of lexical substitution. See Quine (1970/86: 53-55). This last point is controversial as it has been indicated by Boolos (1975: 52-53).

3.5. The late Tarski: current model theory

During the fifties Tarski developed his set-theoretic viewpoint of logical consequence. In this non-fixed universe account, interpretations are set-theoretical objects, namely, elements of the universe of pure sets. This view presupposes and ontology of sets. Validity or invalidity of a given argument-text in this contemporary approach amounts to the non-existence or to the existence of a set that provides for a countermodel. This set-theoretic view suggests a more involved reductionist move, perhaps even a circular one since logic is traditionally understood as the science underlying all the sciences, including set theory. Thus logical properties appear to be dependent on mathematical principles, such as the axiom of infinity, which is a true proposition, but whose necessity seems to be non-logical.

One way out of this dilemma was suggested in Corcoran (1973) and Sagüillo (1997). We may think that Tarski provided not with an explication but rather with a mathematical analogue, or a model, in the engineering sense of the word, of logical consequence, undoubtedly fruitful for mathematical purposes and from the perspective of what Tarski years ago had called scientific semantics. This viewpoint opens up several important issues relating the model, the thing being modeled, and that, in virtue of which the model is a model of something. These important issues shall be taken up again in connection with the discussion of the modal notions of logical implication.

3.6. Material adequacy and modality in the transformational conception

At least three problems can be identified in the search of the necessary and sufficient condition for an argument to be valid in all these accounts of the transformational conception of logical implication:

1. Determination of those parts which are transformable or subject to suitable variations. History of logic shows that knowledge of logical form of a given proposition as opposed to its content has pervaded

tradition and has received permanent attention and reappraisal in the light of new advancements in the field. This problem can be called the problem of the determination of *logical form* of a given proposition. Specifically it is desirable not only to have a conventional distinction between logical and extra-logical terms but also a good criterion that can be invoked for such a distinction.

2. Limitations of means of expression in the given language. The level of generality obtained may depend on the richness of the lexicon of the language considered. This can be called the problem of the linguistic *immanence*. The definition of logical implication must be *transcendent* or extra-linguistic.
3. Expressive limitations of the language may also depend on the [fixed] universe of the interpretation considered. This can be called the problem of *locality* of the universe of discourse. The definition of logical implication must be *global* or independent of the universe of the intended interpretation.

It should be clear by now that the model-theoretic conception of logical truth and consequence that is most widely accepted today, and which goes back at least as far as Tarski (1953) diverges substantially from the substitutional conception favored by Quine. Moreover, it also diverges from the account of logical properties provided by Bolzano, Russell, and Tarski's first view in the thirties. To start with the agreements, it should be emphasized at the outset that all these logicians (including Tarski in the thirties but not in the fifties) worked in a logical framework of an interpreted language, having a fixed-universe of discourse. Having said this, Quine is the only first-orderist among them. Implicitly or explicitly, Bolzano, Russell and Tarski worked within a higher-order framework. On this score, it is worth restating that Bolzano took propositions as the bearers of truth and falsity and his discussion of logical properties involved a non-defined operation of substitution of concepts or ideas within propositions. In a word, his account abstracts from peculiarities of concrete languages. On the contrary, it is well known that Quine would not endorse a discourse committed to propositions like Bolzano's.

Russell (1903/37 and 1919/93) considered the formal implication viewpoint of validity. A given argument-text is formally valid if and only if the corresponding universalized conditional is materially, or truth-functionally true. Certainly, Quine would not approved of a similar characterization of his conception, but it is tempting to think of his definition as even with Russell's in ignoring argument-texts with infinitely many premises but uneven in taking a first-order language instead of a language of types as the object-language.

Tarski's original approach involves an enlargement of the generalization obtained by means of his semantic devices far superior to any of the preceding definitions. Besides he deserves the credit for being the first who clearly indicated what terms to vary in order to characterize the *proper* concept of logical truth and consequence, namely, all the non-logical terms. In this sense, he superseded Bolzano, who discussed different logical properties and relations with respect to different classes of varying ideas. Finally, It should also be emphasized that Tarski (1986) is the only one of these logicians who has provided with an account of the logical notions; namely, his well-known criterion of invariance under permutations of the universe onto itself.

It remains to be discussed the nature of the necessity involved in logical consequence. Particularly, whether any of these accounts within the transformational conception incorporates in some sense any modal notion. This is a difficult point requiring perhaps historical and philosophical sensitivity. It is the opinion of the author that logical truth and logical consequence are non-modal, or rather *de-modalized* in all the approaches considered. The case of Bolzano may be more difficult to assess given the fact that he took propositions to be the bearers of truth. However no account of modality is presented in his book. Once more, each of these characterizations is made purely in terms of generality. In this sense, Tarski seemed to have been with Quine in the well-known avoidance of modal notions of the second. I have no evidence in the direction of an elaborated or explicit modal conception as far as Russell is concerned either. Thus, to restate the point, all these approaches are qualified as reductionist in the sense that logical truth and consequence amounts to truth or preservation of truth under *every* suitable transformation for the content terms. Each of the above viewpoints attempts to define logical truth and consequence by means of an increasing level of generality resorting to one or another kind of variations performed upon certain entities, whether concepts in a given proposition or expressions in a given sentence. Logical truth and consequence then is predicated of the corresponding bearer if it remains true under *every* variation of the kind envisage by the author, whether extra-linguistic or intra-linguistic.

The topic neutrality presupposed by the logical properties so characterized is endangered by analogous conditions: immanence and locality. An immanent characterization *à la* Quine obtains more generality by expanding the language considered, thus rendering it transcendent. A local characterization *à la* Tarski in the thirties obtains more generality by allowing changes in the universe of discourse, thus rendering it global. Likewise, the level of generality of the current model-theoretic account rests heavily on the ontology of pure sets.

It is tempting to think that Tarski and Quine would regard modal notions as mere vestiges of a psychological attitude. Although there are different levels of generality as it was already indicated, generality *prima facie* is something we seem to understand better than modality. The open question is what is special or specific about the generality involved in the modal component of logical truth and consequence. A subsidiary question (but not less important) is in what sense if any, an [extensional] mathematical logic actually incorporates our previous modal intuitions. There are some interesting suggestions in that direction in García-Carpintero (1993 y 2001) and Shapiro (1998). One way or another, these articles suggest that our model-theoretic semantics represents or captures some important features related to our pre-formal modal notions of logical truth and consequence. For views granting the modal feature of our preformal conceptions but not detecting it in the Tarskian semantics see Gómez-Torrente (1996) and Sagüillo (1997).

In a more philosophical fashion, one would like to say that modality is supervenient on generality, or induced by generality, in all these approaches. Focussing on the *alleged* modal component of our current model-theoretic definition of logical consequence, modal supervenience is obtained *modulo* the representational power of current [extensional] mathematical logics. In a strict sense, model-theoretic logical consequence is non-modal, or explicitly de-modalized in current mathematical analogues of logic. However, a sort of *ersatz* modal feature is invoked reflecting on the level of generality obtained. Is this intended modal feature of model theory more than merely rhetorical or it is just debatable hermeneutics for that matter? Perhaps it is useful at this point to recall the suggested hypothesis that model theory provides with a mathematical model or analogue of logical consequence. In a nutshell, model theory does not say what logical consequence is but rather what logical consequence *is like*. As it has been already emphasized the mathematical material out of which the analogue provided by model theory is built is extensional and set-theoretic. Specifically, in this framework considering *all* transformations simply means considering all admissible set-theoretic *interpretations*. In the final analysis the definition of logical properties proposed here rests on what sets there are according to the background set theory used in the logico-mathematical framework. In this sense, relying on set-theoretic principles such as the axiom of infinity, is simply to allow the analysis or definition of a logical property to rest on the material ground of a [true] mathematical proposition. Many unsatisfied logicians would contend however, that the realm of mathematical possibility is narrower than the realm of logical possibility and hence, this maneuver would provide at most with a surrogate: briefly stated, mathematical necessity is not logical necessity.

4. Approaches within the Informational Conception

The Informational conception is connected with the logical positivist emphasis on tautologies as propositions devoid of information and on tautological processes of reasoning as ones that do not add information to the premise-set. The fundamental *postulate* of this conception can be put as follows: **P** logically implies **c** if and only if the information in **P** contains the information in **c**. In this sense, if **P** implies **c,** it would be – strictly speaking – redundant (although perhaps useful) to assert **c** in a context where the propositions in **P** have already been asserted; i.e., no information would be added by asserting **c**.

This conception is pedagogically useful and has natural appealing. Very often, introductory courses in logic and argumentation theory begin with the unqualified statement that deducing is simply unpacking, unfolding or extracting the information more or less hidden in the premises. The following examples point out that the intuition behind the metaphor of information is present in our colloquial and professional argumentative practices.

Perhaps the simplest way to drop information is by eliminating a conjunction. Analogously, perhaps the simplest way to add information is by introducing a conjunction. Disjunction introduction is another usual way of dropping information or avoiding commitment to what was already asserted. In effect, taking something – but not all – back, is what is obtained in asserting the more cautious "I shall visit you in March or April" after having said the more contentful "I shall visit you in March". More dramatically, this maneuver is made more evident when going from "I love you" to "I love you or perhaps not quite so", which – most probably – lacks information. Another simple way of dropping information is by means of a conditional assertion introducing a qualification upon what it was already asserted. For example, when passing from "I parked in the faculty parking lot" to "If I came by car, I parked in the faculty parking lot". It is easily seen that even less information – in fact no information at all – is conveyed by "If I parked in the faculty parking lot, I parked in the faculty parking lot". Other standard procedures of *saying less* or *saying more* are obtained by using restrictive and attributive relative clauses. Thus in the universe of natural numbers, "every oblong number is even" clearly follows from "Every number is even" but "every number, which is inductive, is zero or positive, does not follow from "Every number is zero or positive".

This conception presupposes the existence of information, something whose ontological status is to be distinguished from that of logical implication, from propositions and from the propositions related by logical implication. Distinguished logicians such as Boole (1847), De Morgan (1847), Jevons (1870), Venn (1881/1971), Carnap (1947/60) and Cohen & Nagel (1962/93) made explicit or implicit use of it[1]. More recently, Corcoran (1995, 1998 and 1999) has developed his information-theoretic logic to be discussed shortly. Before that however, it is useful – by way of comparison – to illustrate this conception by surveying first, Carnap and Bar-Hillel's semantic information theory (1952), which builds up from previous work of Carnap (1942/75, 1947/60 and 1950/67).

4.1. Carnap and Bar Hillel extrinsic approach

In outline, Carnap and Bar Hillel envisioned an *explication* for the pre-systematic notion of information content by defining the information content of an interpreted sentence or *statement* to be the class of possible states of *the* universe which are excluded by the given statement. In other words, the class of possible states of the universe, in which the given statement is false *provides* its information content. In their construction, Carnap and Bar Hillel took possible states of the universe as the designata of their state-descriptions. A state-description with respect to a given language is a conjunction that contains for each elementary pair composed of an atomic sentence and its negation, one and only one of its components. Notice that on this account, a tautology has minimum information and a contradiction has maximum information, since a tautology is true in every state and hence it excludes none, and a contradiction is false in every state and hence it excludes all. Hence, under this standpoint of information content it is natural to hold that **P** logically implies **c** if and only if the class of state descriptions in which **c** does not hold is contained in the class of state-descriptions in which **P** does not hold[2].

[1] For detailed relevant quotations see Corcoran (1998).

[2] There is a second route proposed by Carnap (1942/75, 152) that takes the information content of a *sentence* to be the class of its sentential consequences or sentential implications. This second approach -Carnap indicates- belongs to the realm of *expressions* as opposed to the previous one that belongs to the realm of *designata*. However it is clear by the context that Carnap is thinking of an interpreted language in this second option as well. Further research -developed by Carnap himself and Carnap & Bar Hillel jointly-, seems to indicate preference for

It is the opinion of the author that although current possible world semantics involves a natural transformational account of logical implication, Carnap and Bar-Hillel's states account is not transformational. This is due to the fact that their insight is not to re-interpret the language, but rather canvassing ways *the* world could have been according to the means of expression of the [interpreted] language under consideration. In other words, the interpretation of the language is kept fixed and the truth-value of a given sentence is determined with respect to alternative states the world. This suggests qualifying Carnap and Bar Hillel's informational account as *extrinsic*. Notice, that *truth in a state* is clearly an extrinsic property of a sentence in the sense that not only depends on what the sentence says but also on how the state is. Also notice that logical implication based on information content in the present sense is *induced* on truth values, since it is characterized by the inclusion of the class of states in which the conclusion is false in the class of states in which the premises-set is false. It is worth emphasizing once more that this relation so defined rests on something extrinsic to the interpreted sentences, namely, states, and more to the point, it is constructed or represented on the bases of the pure extensional subclass relation. If the class of designata of state-descriptions is *envisioned* as providing for ontological possibilities then it is straightforward to obtain a modal *reading* of logical implication so defined. *Necessarily* if the premises are all true the conclusion is true; i.e., in *every* state in which the premises are true, the conclusion is also true. Similarly, it is *impossible* for the premises to be all true with the conclusion false: i.e., *there is no* state in which the premises are all true and the conclusion is false[3].

4.2. Corcoran's intrinsic approach

Corcoran's information-theoretic logic takes propositions pertaining to a fix universe as containers or *carriers* of information. Propositions on this account are abstract entities, each having its own singularity and complexity. From a purely ontic viewpoint, the information-containment conception of logical implication does not rest in actual truth-values nor in possible truth-values of the propositions involved. Logical validity is an *intrinsic* property of an argument on this conception to the extent that what

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the first *explicatum* in the realm of designata over the second proposal in the realm of expressions.

[3] A similar modal reading of a non-informational approach to logical implication seems to be what Etchemendy (1990, Chapter 2) proposes under the label "representational semantics". It is important to emphasize that this viewpoint presupposes a fixed universe conception as well as a fixed interpretation.

determines it is the information *contained* in the propositions involved and nothing outside the given argument is required.

In addition to the mentioned fundamental postulate, the following remarks characterize this information-theoretic approach to logic: 1. A proposition **c** is independent of a given premise-set **P** if **c** contains information beyond the information contained in **P**. 2. A proposition is tautological if it lacks content. 3. A proposition is contradictory if it contains all information of the universe considered. 4. A proposition and its own negation do not share content. 5. The information contained in a disjunction is the information shared by its members[4].

It is easy to see that under Corcoran's information-containment conception a tautology follows from any premise-set since – lacking information – it can never add information to the one already contained in the premise-set. Likewise a contradiction logically implies any proposition since it contains all the information pertinent to the universe considered.

In a sense, the information-containment conception of logical implication is also *modal* since if a given proposition contains all the information contained in another it is *impossible* for this not to be so. In this intrinsic characterization the identity of the implying proposition and the implied proposition are both involved. Likewise, information in this conception is what logically equivalent propositions have in common. Of course, having the same information content neither entails nor precludes having the same logical form and conversely[5].

Working in the framework of Gödel's arithmetic and based on his conception, Corcoran (1995) has identified the concept of *informational atom* and the concept of *informational saturation*. An [informational] atom is an informative (non-tautological) proposition from which no information can be dropped without render it devoid of content. Notice that an [informational] atom is not a [structural] atomic proposition, since in a standard universe, say the universe of arithmetic propositions, an atomic proposition always logically implies a non-tautological disjunction. In this universe, any two atomic propositions, although being logically independent, they nevertheless share content. In other words, these two are *connected*. For example, "two is even" and "two is prime", each implies their [non-tautological] disjunction. By contrast an informational atom only implies itself and only superimplies tautologies, where 'superimplies' means "implies but it is not implied by". Likewise an informational atom is

4 See Corcoran (1998) and Scanlan & Shapiro (1999).

5 See Corcoran (1998, 116)

a proposition that is logically implied by every proposition that is connected to. Notice that every two non-equivalent atoms are unconnected[6].

In this informational setting, a saturation is a proposition that contains maximal information with respect to a given universe. Thus, a saturation implies every proposition that it does not contradicts or whose negation it does not imply. In other words, any information added to the information of a saturation renders the set of propositions so obtained contradictory or inconsistent. Corcoran (1995: 4.3) points out that in every propositional domain closed under negation, the negation of a given saturation is an atom and the negation of a given atom is a saturation. In this sense, the second-order Gödel axiom-set of arithmetic (G) is a saturation about the class of natural numbers. Likewise, the negation of the Gödel axiom-set is its corresponding atom[7]. Thus, the information-theoretic framework provides with a distinctive way of re-stating fundamental logical concepts, such as, [semantic] completeness and categoricity of a premise-set. Also an important cluster of specific problems can be addressed and newly examined, such as the identification of the *redundancy* of an [informationally] *excessive* premise-set with respect to a given conclusion and the *completion* of an [informationally] *insufficient* premise-set with respect to a given conclusion.

5. Concluding remarks

It is important to bear in mind that the classification into two conceptions of the approaches to logical implication presented in this paper is not intended to indicate the existence of two rival viewpoints on the issue. However, the transformational conception and the informational conception exhibit important differences in the ontic characterization of

6 If considering propositions pertaining to a small universe, a structural atom and an information atom may coincide. For an interesting example, see Corcoran (1995, 74-75).

7 The negation of G is an informational atom since it does not superimply any proposition that is not tautological. Two propositions are unconnected (do not share any information) if and only if their disjunction is tautological. To see that the negation of G is an informational atom it is sufficient to notice that the disjunction of the negation of G with any proposition A is tautological. Since G is a saturation either G implies A or G implies the negation of A. If G implies A then it follows that the disjunction of the negation of G with A is tautological. If G implies the negation of A then it is clear that the disjunction of the negation of G with the negation of A is also tautological.

logical implication as well as in the account of our epistemic access to it. The suggestion here is that these two main conceptions are rather complementary if viewed from the two most important epistemic problems in logic: the problem of determining that a premise-set logically implies a given conclusion when it does, and the problem of determining that a premise-set does not logically implies a given conclusion when it does not. Tradition in classical logic reflecting on the practice of mathematics shows deduction and counter-transformation as the methods *per excellence* to determine respectively, validity of a given valid argument and invalidity of a given invalid argument. How these methods were realized depended pretty much on philosophical taste of the author and mathematical techniques available at the time.

This situation opens up the question for the nature of our knowledge of validity and invalidity. What is the primary source of this knowledge: transformations or information content? In other words, what is the best conception to ground our capabilities to determining validity and invalidity of a given argument? As a matter of fact, each conception of logical implication bears fundamental differences on what is required for these capabilities to explain our knowledge of validity and invalidity[8].

The possibility of deducing seems to fit well the postulation of a mind endowed with certain capabilities to extracting information of a [grasped] proposition or set of propositions and to comparing the information contained in each of the propositions in the *chain* of reasoning leading to the conclusion. In this sense, the information-containment conception only postulates processing the information contained in the propositions without requiring knowledge of whether the information is accurate; i.e. knowledge of logical implication appears in this conception to be a purely conceptual matter or a priori, as it was traditionally thought[9].

The transformational conception may appear philosophically problematic, particularly when considering the way this conception is *captured* in logical systems whose formal language goes beyond the expressive power of first-order. In effect, when deductive completeness is not available, knowledge of validity seems to rest on an rather unfeasible capability of surveying all pertinent structures in order to check that none of them is such that makes the premises all true and the conclusion false.

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8 This fundamental issue is raised in Corcoran (1998, section 6.4.1.). I am in agreement with the important points stated there.

9 See Corcoran (1998, section 6.4.2.)

However knowledge of invalidity of a given argument or knowledge of logical independence of a proposition with respect to a given premise-set, makes the case more plausible for the transformational conception to the detriment of the information-containment conception. Determination of invalidity by a counter-transformation involves two sub-methods: the method of fact and the joint method of form and fact. Invalidity is established by the method of fact when the premises are all known to be true and the conclusion is known to be false. However in other cases, when knowledge of truth-value of the propositions involved is not available, establishing invalidity requires using **PF** since every argument in the same logical form as an invalid by fact argument is also invalid. Solving the problem here means finding and argument in the same logical form of the given argument but with all true premises and false conclusion. Notice there is no way to use the method of form alone to determine invalidity of a given argument. Of course, there are arguments known to be invalid by fact. However, in order to use the method of fact it is necessary but not sufficient to have all true premises and false conclusion. Invalidity of a given argument is not determined by fact if the premises are not *known* to be true or the conclusion is not *known* to be false. Thus, knowledge of invalidity in the transformational conception is founded in many cases on a posteriori knowledge and hence it is not a merely conceptual matter[10].

In order to have a clear sense of the difficulty here, it is tempting to pursue the issue of determining invalidity according to the informational conception. Here is an analogous principle to **PF** that can be called principle of content (**PC**):

> **PC**: Two arguments whose premise-sets and conclusions have respectively the same information content are both valid or both invalid.

It follows from **PC** that every argument with the same information content in the premise-set and with the same information content in the conclusion of an invalid argument, is also invalid. In other words, every argument with premise-set and conclusion respectively [logically] equivalent to the premise-set and to the conclusion of an invalid argument, is also invalid[11].

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[10] Compare Corcoran (1998, *ibidem*)

[11] See Corcoran (1998, section 6.4.1.)

It is easy to see that **PC** is not sufficient to determine invalidity since the principle does not work beyond those invalid arguments in the category of all true premises and false conclusion. The informational conception seems to run short of solutions for invalid arguments in the category of all true premises and true conclusion, or in the category of at least one false premise and true conclusion, or in the category of at least one false premise and false conclusion. It is easy to see that in each of these three cases, determining invalidity of an invalid argument by means of **PC** is logically impossible.

There are further and not less difficult philosophical questions that can be raised on this issue. It seems unfair to think that knowledge was lacked until a particular method of obtaining knowledge is properly regimented, modeled or captured in a logical system. Particularly if current mathematical logic reflects on mathematical practice, it seems that – let us say – pre-formal knowledge of validity and invalidity was already available in order to make further systematic study of it possible. How could otherwise, previous practice warrant or guide our current formal methods of validation and invalidation? Granted this point, it seems fair to contemplate that well-founded practice of establishing invalidity in some cases perhaps does not require a counter-transformation. Let us consider for example the universe of discourse of the natural numbers. It prompts to mind with distinct clarity that the information contained in “two is even” goes beyond the information contained in “three is prime”. Similarly, the premise-set composed of “two is even”, “three is even”, “four is even” and so forth, has less information than the information contained in the corresponding universal closure “every number is even”. Is this merely heuristics as a result of which just responsible judgement is obtained, or is it rather apodictics with the strength of conclusive judgement or knowledge of invalidity for that matter what it is actually gained? The point is that there are simple cases in which comparing information-containment seems available and evident prompting a natural response of the mind[12].

Let me to conclude by summarizing the main points of the present survey. Two main conceptions of logical implication, the transformational conception and the informational conception have been addressed, and some fundamental approaches within each of these conceptions have been articulated. The views of Bolzano, Russell, Tarski in the thirties, Quine, and current model theory furnish with various realizations of the transformational conception of logical implication. The extrinsic view of Carnap and Bar-Hillel on the one hand, and the intrinsic view of Corcoran

[12] Compare Corcoran (1998, *ibidem*)

on the other, were seen as different realizations of the informational conception. Carnap and Bar-Hillel's account was characterized as extrinsic since information content is induced on truth-values at a given state of the world. On the contrary, Corcoran's information-containment approach was characterized as intrinsic since propositions are the carriers of information and nothing outside a given valid argument is relevant for it to have this property. It was argued that both conceptions – the transformational and the informational – rest on alternative insights into the nature of logical implication and into the nature of our epistemic access to this phenomenon. The underlying processes of validation of a given valid argument and invalidation of a given invalid argument seem to indicate that both conceptions have played a fundamental role in concocting our current comprehension of logical practice. This is not to say that we have a full understanding of logical implication, much less that we have a comprehensive and unique source of this partial understanding.

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