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TWO LOGICS OF ANALYTIC CLASSICAL IMPLICATION

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Abstract

The paper contains two concepts of implication that can be called *analytic* in the sense of Parry [5] or Fine [2]. Contrary to the Parry's approach, these implications do not involve *S4* strict implication but the classical (material) one. This fact refers to the similar notion of so-called *demodalized analytic implication* of Dunn [1] and *strong implication* of Vanderveken, cf. [4, 6]. The features of analytic classical implications are presented in the form of two propositional logics on the pure implicational language.

1. Preliminaries

By a *generalized Kripke model for a propositional language* $\mathsf{L} = (L, F_1, ..., F_n)$ (cf. [3]) we mean any pair $m = \langle W_m, 1_m \rangle$ such that W_m is any nonempty set (of points) and $1_m \subseteq W_m \times L$ is any binary relation (of satisfiability of a formula at a point).

Given a class M of generalized Kripke models for the language L, the consequence relation 1^{M} defined for any set of formulas $X \subseteq L$ and a formula $\alpha \in L$ as follows:

$$X \; 1^{\mathsf{M}} \alpha \quad \text{iff} \quad \forall\, m \in \mathsf{M} \;\; \forall\, x \in W_m, \;\; x \; 1_m \alpha \text{ whenever } \forall\, \beta \in X, x \; 1_m \beta,$$

will be called the *consequence determined by* M.

The semantics of generalized Kripke models turns out to be of general type as it is shown in the following elementary result (cf. [3]):

for any consequence relation 0 *on the language* L*, no matter if finitary or structural, there exists a class* M *of generalized Kripke models for* L *such that* $0 = 1^{\mathsf{M}}$*, namely,* M *is a singleton composed out of the model* $m = \langle Th(0), 1_m \rangle$*, where* $Th(0)$ *is the family of all theories of* 0 *and* 1_m *is the converse relation of* $\in$*, i.e, for an* $X \in Th(0)$ *and* $\alpha \in L$*,* $X \; 1_m \alpha$ *iff* $\alpha \in X$.

In a general approach there is no constraint put for the relation 1_m in a model *m*. However, in particular cases the relation is determined by some parameters, often having an extralinguistic character. For example, in a usual Kripke model for a given normal modal logic two parameters: a relation of accessibility and a valuation providing a logical value at a point to any propositional variable, are indispensable to determine the satisfiability relation. The latter even do not appear explicitly in a Kripke model, contrary to those two parameters.

Usually only such classes $\mathbb{M}$ are taken into account that each of them is composed out of the *similar* generalized Kripke models, similar in the sense that in each model $m \in \mathbb{M}$ the same parameters are used to determine the relation 1_m in some uniform way.

First, we will apply a semantics of generalized Kripke models to define a logic of analytic strict implication on the pure implicational propositional language $\mathbb{L} = (L,\to)$, i.e. with the only 2-ary connective $\to$ (of implication), freely generated by the set *Var* = $\{p_0,p_1,...\}$ of propositional variables. The formal concept of analytic implication was introduced by Parry in [5], where he defined a logic (of analytic implication) as an axiom system on the propositional language with the connectives of negation, conjunction and analytic implication, having the following property: if an implication $\alpha \to \beta$ belongs to the system, then it is a theorem of *S4* system of strict implication and each propositional variable occurring in the formula β also occurs in α. After Dunn [1] we will call that system a logic of analytic strict implication. In [1] Dunn strengthened the system by the following two axioms: $(\alpha \wedge \beta) \to {\sim}(\alpha \wedge \beta)$ and $\alpha \to ({\sim}\alpha \to \alpha)$. The last one together with *S4* logic of strict implication results as the classical logic. Hence, in the new system of Dunn, the analytic implication is no longer strict but classical. This yields the following concept of analytic implication as a relation of inference (cf. [1]): a proposition *P* analytically implies a proposition *Q* iff the content of *Q* is included in the content of *P*, and furthermore, *P* is not true while the proposition *Q* is false. The same idea of analytic implication is due to Vanderveken (cf. [4, 6]), however under the name of *strong implication* (there are some differences concerning the concept of *content of a proposition* in both authors). Fine [2] provided a Kripke style semantics for the system of Parry, truly speaking for its slight extension (by the axiom $(\alpha \wedge {\sim}\beta) \to {\sim}(\alpha \to \beta)$ - cf. also [1]) which from the point of view of its semantics should be actually called a logic of analytic strict implication.

Now, the implicational part of this logic can be defined semantically after Fine [2] in terms of generalized Kripke models. Let M_a be a class of all models $m = <W_m, 1_m>$, where the relation 1_m is determined by the following parameters:

- a reflexive and transitive relation $\Xi \subseteq W_m \times W_m$,
- a family $\{(C_x, \vee_x)$: $x \in W_m\}$ of semilattices with the orderings: for any $x \in W_m$ and any $a,b \in C_x$, $a \leq_x b$ iff $a \vee_x b = b$,
- a family $\{f_x$: $x \in W_m\}$ of homomorphisms f_x: $(L, \rightarrow) \longrightarrow (C_x, \vee_x)$,
- a function v: $Var \longrightarrow P(W_m)$.

The definition of 1_m is as follows. For any $x \in W_m$, $p \in Var$, $\alpha, \beta \in L$,

$x\ 1_m\ p$ iff $x \in v(p)$,
$x\ 1_m\ \alpha \rightarrow \beta$ iff $\forall y \in W_m$, if $x\ \Xi\ y$, then $((y\ 7_m\ \alpha$ or $y\ 1_m\ \beta)$ and $f_y(\beta) \leq_y f_y(\alpha))$.

In what follows, the consequence relation $1^{\mathsf{M}a}$ will be called a logic of analytic strict implication.

However, using the same four parameters one can define the other relation of satisfiability: $1_{m'}$, obtaining the other class, say, $\mathsf{M}_a{}'$ of all the models $m' = <W, 1_{m'}>$ such that the consequence relation $1^{\mathsf{M}a}{}'$ might be also called a logic of analytic (*S4*) strict implication. Namely, for given nonempty set of points W, a reflexive and transitive relation $\Xi \subseteq W \times W$ and a function v: $Var \longrightarrow P(W)$, let 1_{S4} be a relation of satisfiability defined as in a Kripke model for *S4* logic of strict implication:

$x\ 1_{S4}\ p$ iff $x \in v(p)$,
$x\ 1_{S4}\ \alpha \rightarrow \beta$ iff $\forall y \in W$ (if $x\ \Xi\ y$, then $y\ 7_{S4}\ \alpha$ or $y\ 1_{S4}\ \beta$).

Then, given a set of points W and those four parameters: Ξ, $\{(C_x, \vee_x)$: $x \in W\}$, $\{f_x$: $x \in W\}$, v, let us define a relation of satisfiability $1_{m'}$ as follows:

$x\ 1_{m'}\ p$ iff $x \in v(p)$,
$x\ 1_{m'}\ \alpha \rightarrow \beta$ iff $x\ 1_{S4}\ \alpha \rightarrow \beta$ and $f_x(\beta) \leq_x f_x(\alpha)$,

where the relation $1_{S4} \subseteq W \times L$ is determined by the parameters Ξ and v.

Denote by $\mathsf{S4}$ the set of all theorems of the pure implicational part of *S4* logic of strict implication, and for any formula α let $V(\alpha)$ be the set of all propositional variables occurring in α. A fundamental property of the just defined logic of analytic strict implication as well as a mutual dependence between two logics of analytic strict implication can be noticed as follows.

Proposition 1.1. For any formulae α, β:

(1) $1^{M_a'}\ \alpha \to \beta$ iff $\alpha \to \beta \in$ S4 and $V(\beta) \subseteq V(\alpha)$.

(2) $1^{M_a}\ \alpha$ implies that $1^{M_a'}\ \alpha$ but, in general, not conversely.

Proof. (1)($\Rightarrow$): Consider a class $M_0 \subseteq M_a'$ of all the models determined by the same 1-element family of semilattices composed out of 1-element semilattice $(\{1\},\vee)$ and, consequently, by 1-element class of homomorphisms composed out of the homomorphism from the language onto that semilattice. Then obviously, $1^{M_a'} \subseteq 1^{M_0}$ and $\{\alpha \in L\colon 1^{M_0}\ \alpha\} =$ S4. In order to show that $V(\beta) \subseteq V(\alpha)$ whenever $1^{M_a'}\ \alpha \to \beta$, consider whatever model $m' \in M_a'$ determined by the 1-element family of semilattices composed out of the following semilattice of sets: $(P(Var),\cup)$, and by 1-element class of homomorphisms composed out of the homomorphism V of $(L,\to)$ into $(P(Var),\cup)$. Then in such a model for any its point x we have: $x\ 1_{m'}\ \alpha \to \beta$ iff $x\ 1_{S4}\ \alpha \to \beta$ and $V(\beta) \subseteq V(\alpha)$ (where the relation 1_{S4} is determined by the parameters Ξ and v of the model m'). This yields the proved implication.

(1)($\Leftarrow$): Assume that $\alpha \to \beta \in$ S4 and $V(\beta) \subseteq V(\alpha)$. Take any model $m' \in M_a'$ and any its point x. From the first assumption it follows that $x\ 1_{S4}\ \alpha \to \beta$. From the second one and the fact that for any formula γ: $f_x(\gamma) = f_x(p_1) \vee_x ... \vee_x f_x(p_n)$, where $\{p_1,...,p_n\} = V(\gamma)$ and f_x is a homomorphism of $(L,\to)$ into a semilattice $(C_x,\vee_x)$, it follows that $f_x(\beta) \leq_x f_x(\alpha)$. Finally, $x\ 1_{m'}\ \alpha \to \beta$.

(2): Since $1^{M_a}\ p$, where $p \in Var$, never holds, assume that $1^{M_a}\ \beta \to \gamma$ for some $\beta, \gamma \in L$. According to (1) we have to show that $\beta \to \gamma \in$ S4 and $V(\gamma) \subseteq V(\beta)$. This follows by the same argument used in the proof of (1)($\Rightarrow$). To the end notice that for example for the formulae α, β of the form, respectively: $(p \to q) \to (p \to q)$, $q \to (p \to p)$, we have that $\alpha \to \beta \in$ S4 and $V(\beta) \subseteq V(\alpha)$, therefore $1^{M_a'}\ \alpha \to \beta$ by (1). However, $7^{M_a}\ \alpha \to \beta$. For example, in any 1-element model $m = \langle\{x\},1_m\rangle \in M_a$, in which the relation 1_m is determined by the semilattice $(P(Var),\cup)$ and the homomorphism V, the following holds: $x\ 1_m\ \alpha$ and $x\ 7_m\ \beta$.

Below, *per analogiam* to two logics of analytic strict implication, two logics of analytic *classical* implication are defined.

2. First logic of analytic classical implication

The first logic of analytic classical implication we are going to consider is determined by the class M_1 of all 1-element generalized Kripke models $m = <\{x\}, 1_m>$, where x is any point and the relation 1_m is determined by the following three parameters:

- any semilattice $(C,\vee)$ with the semilattice ordering $a \leq b$ iff $a \vee b = b$,
- any homomorphism f of $(L,\to)$ into $(C,\vee)$,
- any function v: $Var \longrightarrow P(\{x\})$,

dependently upon the length of a formula, in the following way (instead of $x\ 1_m\ \alpha$ we simply write $1_m\ \alpha$, any $\alpha \in L$):

$1_m\ p$ iff $v(p) = \{x\}$, any $p \in Var$,
$1_m\ \alpha \to \beta$ iff $(7_m\ \alpha$ or $1_m\ \beta)$ and $f(\beta) \leq f(\alpha)$.

Let us define the new binary connective $\Rightarrow$ as follows:

$$\alpha \Rightarrow \beta = ((\alpha \mapsto \beta) \to \alpha) \to ((\alpha \mapsto \beta) \to \beta),$$

where $\alpha \mapsto \beta = (\alpha \to \beta) \to (\alpha \to \beta)$. It is easy to check that the truth condition for a formula $\alpha \Rightarrow \beta$ in any model $m \in \mathsf{M}_1$ is of the form:

($\Rightarrow$) $1_m\ \alpha \Rightarrow \beta$ iff $7_m\ \alpha$ or $1_m\ \beta$.

Let 1_{cl} be the pure implicational part of the classical logic (on the language $(L,\to)$). Then we have

Proposition 2.1. For any formulae α, β*:*

(1) $1^{\mathsf{M}_1}\ \alpha \to \beta$ iff $1^{\mathsf{M}_1}\ \alpha \Rightarrow \beta$ and $V(\beta) \subseteq V(\alpha)$,

(2) $1^{\mathsf{M}_1}\ \alpha \to \beta$ implies $1_{cl}\ \alpha \to \beta$ (in general, $1^{\mathsf{M}_1} \subseteq 1_{cl}$),

(3) in general, the conditions: $1_{cl}\ \alpha \to \beta$, $V(\beta) \subseteq V(\alpha)$ do not imply that $1^{\mathsf{M}_1}\ \alpha \to \beta$.

Proof. (1)($\Rightarrow$): The fact that $1^{\mathsf{M}_1}\ \alpha \to \beta$ implies $1^{\mathsf{M}_1}\ \alpha \Rightarrow \beta$, is obvious due to the truth condition ($\Rightarrow$). Now assume that $1^{\mathsf{M}_1}\ \alpha \to \beta$. In order to show that $V(\beta) \subseteq V(\alpha)$ consider a model $m \in \mathsf{M}_1$ such that the relation 1_m is determined by the semilattice of sets $(P(Var),\cup)$ and the homomorphism V: $(L,\to) \longrightarrow (P(Var),\cup)$ assigning to each formula the set of all propositional variables occurring in it. Then the result follows from the assumption and the following truth condition of a formula $\alpha \to \beta$ in such a model: $1_m\ \alpha \to \beta$ iff ($7_m\ \alpha$ or $1_m\ \beta$) and $V(\beta) \subseteq V(\alpha)$.

(1)($\Leftarrow$): Assume that $1^{M_1}\ \alpha \Rightarrow \beta$ and $V(\beta) \subseteq V(\alpha)$, and consider any model $m \in M_1$, determined by the parameters: $(C,\vee)$ (with the partial ordering $\leq$), f and v. Since for any formula γ, $f(\gamma) = f(q_1) \vee ... \vee f(q_n)$, where $\{q_1,...,q_n\} = V(\gamma)$ so $f(\beta) \leq f(\alpha)$ due to the assumption. Moreover, from the assumption and the condition ($\Rightarrow$) it follows that $7_m\ \alpha$ or $1_m\ \beta$. Consequently, $1_m\ \alpha \to \beta$.

(2): In order to show that $1^{M_1} \subseteq 1_{cl}$ consider the class $M \subseteq M_1$ of all the models $<\{x\},1_m>$ with the same point x and the different relations 1_m determined in each model by the same 1-element semilattice $(\{1\},\vee)$, the homomorphism f: $(L,\to) \longrightarrow (\{1\},\vee)$, however by different functions v: $Var \longrightarrow P(\{x\})$. Then obviously, $1^{M_1} \subseteq 1^{M}$. Having in each model $m \in M$ the following truth condition: $1_m\ \alpha \to \beta$ iff $7_m\ \alpha$ or $1_m\ \beta$, one can easily show the equality: $1^{M} = 1_{cl}$.

(3): Consider the formulas α, β, respectively, of the form: $p \mapsto q$ and $p \to (q \to p)$, where p,q are different propositional variables. Then $1_{cl}\ \alpha \to \beta$ and $V(\beta) \subseteq V(\alpha)$. However according to (1), $7^{M_1}\ \alpha \to \beta$, since $7^{M_1}\ \alpha \Rightarrow \beta$: in the model m determined by the semilattice $(P(Var),\cup)$ and the homomorphism V considered in the proof of (1), we have: $7_m\ \alpha \Rightarrow \beta$.

In order to provide a sound and complete axiomatization of the logic 1^{M_1} let us introduce the other 2-ary connective $>$ in the following way:

$$\alpha > \beta = \alpha \to (\beta \to \beta).$$

Then in any model $m \in M_1$:

(>) $1_m\ \alpha > \beta$ iff $f(\beta) \leq f(\alpha)$.

Now we show that the following set of axioms is sound and complete for the logic 1^{M_1}:

(Ax1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,
(Ax2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$,
(Ax3) $(\alpha > \beta) \Rightarrow ((\alpha \Rightarrow \gamma) \Rightarrow (((\alpha \to \beta) \Rightarrow \gamma) \Rightarrow \gamma))$,

(Ax4) $(\alpha \to \beta) \Rightarrow (\alpha \Rightarrow \beta)$,
(Ax5) $(\alpha \to \beta) \to (\alpha > \beta)$,
(Ax6) $(\alpha > \beta) \Rightarrow (\beta \Rightarrow (\alpha \to \beta))$,
(Ax7) $\alpha > \alpha$,
(Ax8) $(\alpha > \beta) \Rightarrow ((\beta > \gamma) \Rightarrow (\alpha > \gamma))$,
(Ax9) $(\alpha \to \beta) > \alpha$,
(Ax10) $(\alpha \to \beta) > \beta$,
(Ax11) $(\alpha > \beta) \Rightarrow ((\alpha > \gamma) \Rightarrow (\alpha > (\beta \to \gamma)))$,
$(\mathrm{MP})_{\Rightarrow}$ $\alpha, \alpha \Rightarrow \beta/\beta$.

Denote by 0_1 the consequence relation determined by the above rules of inference. First notice that

$(\mathrm{MP})_{\to}$ $\alpha, \alpha \to \beta/\beta$

is a rule of 0_1 (due to (Ax4) and $(\mathrm{MP})_{\Rightarrow}$). Next notice that the deduction theorem holds for implication $\Rightarrow$: for any $X \subseteq L$, $\alpha, \beta \in L$,

$(\mathrm{DT})_{\Rightarrow}$ $X\, 0_1\, \alpha \Rightarrow \beta$ whenever $X \cup \{\alpha\}\, 0_1\, \beta$,

since $(\mathrm{MP})_{\Rightarrow}$ is the only nonaxiomatic rule of inference in the set of rules defining the consequence 0_1, and the axioms (Ax1), (Ax2) are in that set.

On the basis of truth conditions ($\Rightarrow$), ($>$), one can easily show that the soundness theorem holds: $0_1 \subseteq 1^{M_1}$.

In order to prove the completeness first consider the family $Max(0_1)$ of all the theories of 0_1 which are relatively maximal (sometimes called *saturated*) with respect to a formula: for any theory X of 0_1, $X \in Max(0_1)$ iff there exists a formula $\alpha \notin X$ such that for every formula $\beta \notin X$, $X \cup \{\beta\}$ $0_1\, \alpha$.

The most useful version of the Lindenbaum lemma just takes into account the notion of a relatively maximal theory:

for any finitary propositional logic 0*, any set of formulas* X *and a formula* α *such that* $X\, 6\, \alpha$*, there exists a* $Y \in Max(0)$ *such that* $X \subseteq Y$ *and* $\alpha \notin Y$.

Now, the most important feature of the relatively maximal theories of our logic 0_1 is as follows:

Fundamental lemma for 0_1. For any $X \in Max(0_1)$ and any formulae α, β*:*
$\alpha \to \beta \in X$ iff $(\alpha \notin X$ or $\beta \in X)$ and $\alpha > \beta \in X$.

Proof. Let X be a relatively maximal theory with respect to a formula γ, that is $\gamma \notin X$ and $\forall\ \delta \notin X$, $X \cup \{\delta\}\ 0_1\ \gamma$.

The implication ($\Rightarrow$) of the lemma is obvious for X is closed on $(MP)_{\rightarrow}$ and (Ax5).

In order to prove the implication ($\Leftarrow$) assume first that $\alpha > \beta \in X$ and $\beta \in X$. Then $\alpha \rightarrow \beta \in X$ immediately by (Ax6) and $(MP)_{\Rightarrow}$. Next assume that $\alpha > \beta \in X$, $\alpha \notin X$ and moreover, $\alpha \rightarrow \beta \notin X$. Then $X \cup \{\alpha\}\ 0_1\ \gamma$ and $X \cup \{\alpha \rightarrow \beta\}\ 0_1\ \gamma$. Applying $(DT)_{\Rightarrow}$ and the fact that X is a theory we have that $\alpha \Rightarrow \gamma$, $(\alpha \rightarrow \beta) \Rightarrow \gamma \in X$. Hence and from the assumption, (Ax3), $(MP)_{\Rightarrow}$ it follows that $\gamma \in X$. That is absurd.

Having that fundamental lemma, the construction of the canonical model for the logic 0_1 is not difficult. Take into account any theory X of the logic 0_1. Then a binary relation $\approx_X$ defined on the set of all formulas L as follows (cf. [2]):

$$\alpha \approx_X \beta \quad \text{iff} \quad \alpha > \beta,\ \beta > \alpha \in X,$$

is a congruence relation of the algebra $(L, \rightarrow)$. Indeed, reflexivity and transitivity follows, respectively, from (Ax7) and (Ax8), $(MP)_{\Rightarrow}$, while the condition: $\alpha \approx_X \beta$ and $\gamma \approx_X \delta$ implies $\alpha \rightarrow \gamma \approx_X \beta \rightarrow \delta$, follows from (Ax8), (Ax9), (Ax10), (Ax11) and $(MP)_{\Rightarrow}$.

In that way the quotient algebra $(L/{\approx_X}, \vee)$ is at our disposal, where the binary peration $\vee$ is defined as follows: $[\alpha] \vee [\beta] = [\alpha \rightarrow \beta]$, any $\alpha, \beta \in L$. The algebra is a semilattice. Indeed, from (Ax9), for any $\alpha \in L$, $(\alpha \rightarrow \alpha) > \alpha \in X$. Moreover $\alpha > (\alpha \rightarrow \alpha) \in X$ due to (Ax7), (Ax11), $(MP)_{\Rightarrow}$. In that way, $[\alpha \rightarrow \alpha] = [\alpha]$, that is $[\alpha] \vee [\alpha] = [\alpha]$. Commutativity follows from (Ax11) in the form: $((\alpha \rightarrow \beta) > \beta) \Rightarrow (((\alpha \rightarrow \beta) > \alpha) \Rightarrow ((\alpha \rightarrow \beta) > (\beta \rightarrow \alpha)))$ and (Ax10), (Ax9), $(MP)_{\Rightarrow}$. In order to prove that the operation $\vee$ is associative notice that $(\alpha \rightarrow (\beta \rightarrow \gamma)) > (\beta \rightarrow \gamma) \in X$ according to (Ax10). Similarly, $(\beta \rightarrow \gamma) > \gamma \in X$ and $(\beta \rightarrow \gamma) > \beta \in X$ due to (Ax9). So applying (Ax8) and $(MP)_{\Rightarrow}$ it can be obtained that $(\alpha \rightarrow (\beta \rightarrow \gamma)) > \gamma \in X$ and $(\alpha \rightarrow (\beta \rightarrow \gamma)) > \beta \in X$. Moreover, $(\alpha \rightarrow (\beta \rightarrow \gamma)) > \alpha \in X$ by (Ax9). Finally, from (Ax11) and $(MP)_{\Rightarrow}$ it follows that $(\alpha \rightarrow (\beta \rightarrow \gamma)) > ((\alpha \rightarrow \beta) \rightarrow \gamma) \in X$. Analogously one can show that $((\alpha \rightarrow \beta) \rightarrow \gamma) > (\alpha \rightarrow (\beta \rightarrow \gamma)) \in X$ which leads to the result: $[\alpha \rightarrow (\beta \rightarrow \gamma)] = [(\alpha \rightarrow \beta) \rightarrow \gamma]$ and, consequently, $[\alpha] \vee ([\beta] \vee [\gamma]) = ([\alpha] \vee [\beta]) \vee [\gamma]$.

Now, for any fixed theory X of the logic 0_1 let $m_X = <\{X\}, 1_{m_X}>$ be a model from M_1 determined by the following three parameters: the semilattice $(L/{\approx_X}, \vee)$, the canonical homomorphism f: $(L,\rightarrow) \longrightarrow (L/{\approx_X},\vee)$ (i.e., for any $\alpha \in L$, $f(\alpha) = [\alpha]$) and the function v: $Var \longrightarrow P(\{X\})$ given as follows: for any $p \in Var$, $v(p) = X$ iff $p \in X$.

Notice that in the model m_X, the condition $[\beta] \vee [\alpha] = [\alpha]$, is equivalent to the following one: $\alpha > \beta \in X$. For assuming the former, we have immediately: $\alpha > (\beta \rightarrow \alpha) \in X$. So taking (Ax9), (Ax8), $(MP)_{\Rightarrow}$ into account, it is easily seen that $\alpha > \beta \in X$. Conversely, from the last expression it follows that $\alpha > (\beta \rightarrow \alpha) \in X$ due to (Ax7), (Ax11), $(MP)_{\Rightarrow}$. Since $(\beta \rightarrow \alpha) > \alpha \in X$ ((Ax10)), so $\beta \rightarrow \alpha \approx_X \alpha$, that is $[\beta] \vee [\alpha] = [\alpha]$.

In that way the truth condition for implicational formula in the model m_X is of the form:

$$1_{m_X} \alpha \rightarrow \beta \quad \text{iff} \quad (7_{m_X} \alpha \text{ or } 1_{m_X} \beta) \text{ and } \alpha > \beta \in X.$$

We are attracted only to the models m_X such that $X \in Max(0_1)$.

Main lemma for 0_1. For any theory $X \in Max(0_1)$ and any formula α, 1_{m_X} α iff $\alpha \in X$.

Proof. Standard with the fundamental lemma for 0_1.

As an immediate consequence of the main lemma and Lindenbaum lemma, the completeness theorem follows: $1^{\mathsf{M}_1} \subseteq 0_1$.

3. Second logic of analytic classical implication

The second logic of analytic classical implication, contrary to the first one, realizes the property: a formula $\alpha \rightarrow \beta$ is its theorem iff it is a classical theorem ($1_{cl}\ \alpha \rightarrow \beta$) and every propositional variable occurring in β also occurs in α.

The class M_2 determining this logic on the languague $(L,\rightarrow)$ is composed out of all the models which differ from those of M_1 by the fact that instead of third parameter: a function v: $Var \longrightarrow \mathrm{P}(\{x\})$, now an arbitrary classical valuation appears, that is a function w: $L \longrightarrow \{0,1\}$ fulfilling the condition: $w(\alpha \rightarrow \beta) = 1$ iff either $w(\alpha) = 0$ or $w(\beta) = 1$. In what follows, the truth conditions in such the models are also modified:

$1_m\, p$ iff $w(p) = 1$,
$1_m\, \alpha \to \beta$ iff $w(\alpha \to \beta) = 1$ and $f(\beta) \leq f(\alpha)$.

Proposition 3.1. For any formulae α, β,

(1) $1^{\mathsf{M}_2}\, \alpha \to \beta$ implies $1_{cl}\, \alpha \to \beta$ (in general, $1^{\mathsf{M}_2} \subseteq 1_{cl}$),

(2) $1^{\mathsf{M}_2}\, \alpha \to \beta$ iff $1_{cl}\, \alpha \to \beta$ and $V(\beta) \subseteq V(\alpha)$,

(3) $1^{\mathsf{M}_1}\, \alpha$ implies that $1^{\mathsf{M}_2}\, \alpha$ (in general not conversely), however $1^{\mathsf{M}_1} \not\subseteq 1^{\mathsf{M}_2}$.

Proof. (1): The proof of inclusion $1^{\mathsf{M}_2} \subseteq 1_{cl}$, is analogous to that of Proposition 2.1(2). So let $\mathsf{M} \subseteq \mathsf{M}_2$ be the class of all models $<\{x\}, 1_m>$ with the same point x and the different relations 1_m determined in each model by the same 1-element semilattice $(\{1\},\vee)$, the homomorphism f: $(L,\to) \longrightarrow (\{1\},\vee)$, however, by different classical valuations. Since $1^{\mathsf{M}_2} \subseteq 1^{\mathsf{M}}$, it is enough to show the equality $1_{cl} = 1^{\mathsf{M}}$. The inclusion $(\subseteq)$ follows from the fact that in each model $m \in \mathsf{M}$, for every formula α, the following truth condition holds: $1_m\, \alpha$ iff $w(\alpha) = 1$, where w is the valuation of m. The converse inclusion is an obvious consequence of the fact that for any classical valuation w there exists a model $m \in \mathsf{M}$ with w as the third parameter and the above truth condition.

(2)($\Rightarrow$): Assume that $1^{\mathsf{M}_2}\, \alpha \to \beta$. Then obviously $1_{cl}\, \alpha \to \beta$ from (1). The fact that $V(\beta) \subseteq V(\alpha)$ can be proved analogously as in the proof of Proposition 2.1(1).

(2)($\Leftarrow$): Similar to the proof of Proposition 2.1(1).

The implication of (3) follows directly from (2) and Proposition 2.1(1),(2). The fact that for some formula α the converse implication: $1^{\mathsf{M}_2}\, \alpha$ implies that $1^{\mathsf{M}_1}\, \alpha$, does not hold, follows from (2) and Proposition 2.1(3). The rule $(MP)_{\to}$ is obviously valid in the logic 1^{M_1} however it is not valid in the second logic (put for example in that rule α and β of the form, respectively, $(p \to q) \to (p \to q)$ and $p \to (q \to p)$) which yields $1^{\mathsf{M}_1} \not\subseteq 1^{\mathsf{M}_2}$.

Now let us define the following unary connective: $\#\alpha = (\alpha \to \alpha) \to \alpha$. Notice that for any formula α in any model $m \in \mathsf{M}_2$:

(#) $\quad 1_m\, \#\alpha$ iff $w(\alpha) = 1$, where w is the classical valuation of the model m. This yields

Lemma 3.2. For any formula α, $1^{M_2} \#\alpha$ iff $1_{cl}\ \alpha$.

We will show that the following set of rules of inference is sound and complete with respect to semantics M_2:

(#A1)	$\#(\alpha \to (\beta \to \alpha))$,
(#A2)	$\#((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$,
(#A3)	$\#(((\alpha \to \beta) \to \gamma) \to ((\alpha \to \gamma) \to \gamma))$,
(Ax7)	$\alpha > \alpha$,
(Ax9)	$(\alpha \to \beta) > \alpha$,
(Ax10)	$(\alpha \to \beta) > \beta$,
(#MP)	$\#\alpha, \#(\alpha \to \beta) / \#\beta$,
(RAx5)	$\alpha \to \beta / \alpha > \beta$,
(RAx6)	$\alpha > \beta, \#(\alpha \to \beta) / \alpha \to \beta$,
(RAx8)	$\alpha > \beta, \beta > \gamma / \alpha > \gamma$,
(RAx11)	$\alpha > \beta, \alpha > \gamma / \alpha > (\beta \to \gamma)$,
(R1)	$\alpha / \#\alpha$,
(R2)	$\#p / p,\ p \in Var$,

(< is defined as before). It is easily seen that the formulae:

(Ax5): $(\alpha \to \beta) \to (\alpha > \beta)$,
$(\alpha > \beta) \to (\#(\alpha \to \beta) \to (\alpha \to \beta))$,
(Ax8): $(\alpha > \beta) \Rightarrow ((\beta > \gamma) \Rightarrow (\alpha > \gamma))$,
(Ax11): $(\alpha > \beta) \Rightarrow ((\alpha > \gamma) \Rightarrow (\alpha > (\beta \to \gamma)))$,
$\alpha \to \#\alpha$,
$\#\alpha \to \alpha$,

are logically true in M_2. However, they are useless as axioms of 1^{M_2}. For neither $(MP)_{\to}$ nor $(MP)_{\Rightarrow}$ are valid rules of that logic (in case of $(MP)_{\Rightarrow}$ consider, for instance, the same example of formulas α, β as in the proof of Proposition 3.1(3); now the relevant truth condition in any $m \in M_2$ is of the form: $1_m\ \alpha \Rightarrow \beta$ iff $w(\alpha) = 0$ or $w(\beta) = 1$, where w is the classical valuation of the model m). For this reason, in the above set of rules, instead of these axioms, the appropriate nonaxiomatic rules of inference occur.

Observe also that $\#\alpha/\alpha$ is not a valid rule of 1^{M_2} (for example, $\{\#(p \to q)\}\ 7^{M_2}\ p \to q$ for different variables p,q). However, (R2) is valid.

Consequently, the consequnce relation, say 0_2, determined by that set of rules is non-structural. However, as it will become clear, the set of its theorems is closed upon any substitution of the language.

It is a routine matter to show that $0_2 \subseteq 1^{\mathsf{M}_2}$. In order to prove the converse inclusion, first let us formulate two helpful propositions.

Fundamental lemma for 0_2. For any theory X of 0_2 and for any formulae α, β: $\alpha \to \beta \in X$ iff $\#(\alpha \to \beta) \in X$ and $\alpha > \beta \in X$.

Proof. ($\Rightarrow$): by (R1), (RAx5).
($\Leftarrow$): by (RAx6).

Lemma 3.3. For any theory X of 0_2, the set of formulae: $\#X = \{\alpha \in L\colon \#\alpha \in X\}$ is a theory of the logic 1_{cl} (of classical implication) such that $X \subseteq \#X$.

Proof. Notice that the formulae (A1), (A2), (A3) (i.e. these ones obtained from (#A1), (#A2), (#A3) by omitting #) together with $(\mathrm{MP})_\to$ form an adequate axiomatics for 1_{cl}. So when X, being a theory of 0_2, is closed on (#A1), (#A2), (#A3), (#MP), the set $\#X$ is closed upon (A1), (A2), (A3), $(\mathrm{MP})_\to$. Thus $\#X$ is a theory of 1_{cl}.

The inclusion given in the lemma follows from the fact that when X is a theory of 0_2, it is closed on (R1).

Now, having the rules (Ax7), (RAx8), (Ax9), (Ax10), (RAx11) one can go along the construction lines of the canonical model for the first logic of classical analytic implication, obtaining for each theory X of 0_2, the semilattice $(L/{\approx_X},\vee)$. So let us consider for any fixed $X \in Th(0_2)$ a bundle M_X composed out of all the models $m = \langle\{X\},1_m\rangle$ from the class M_2 which are determined by the same semilattice $(L/{\approx_X},\vee)$ and the canonical homomorphism f of $(L,\to)$ onto $(L/{\approx_X},\vee)$, and only a classical valuation w differentiates them. Similarly as before, in each model $m \in \mathsf{M}_X$ the condition: $f(\beta) \leq f(\alpha)$, is equivalent to the following one: $\alpha > \beta \in X$. Therefore the truth conditions in such a model are as follows:

$1_m\, p$ iff $w(p) = 1$,
$1_m\, \alpha \to \beta$ iff $w(\alpha \to \beta) = 1$ and $\alpha > \beta \in X$,

where w is the classical valuation of that model.

Taking into account the fundamental lemma for 0_2 we have immediately the following

Lemma 3.4. For any $X \in Th(0_2)$, $m \in \mathsf{M}_X$ and $\alpha \in X$: $1_m\ \alpha$ iff $w(\alpha) = 1$, where w is the classical valuation of m.

The proof of completeness for 0_2 goes as follows. Supposing that $\alpha \notin X$, where a theory $X = \{\beta \in L\colon Y\ 0_2\ \beta\}$ for some set of formulae Y, let us consider two possible cases: $\alpha \notin \#X$, $\alpha \in \#X$. In the first case, it follows from Lemma 3.3 that $\#X\ 7_{cl}\ \alpha$. So apply the Lindenbaum lemma for the logic 1_{cl} obtaining a (relatively) maximal theory Z of 1_{cl} such that $\#X \subseteq Z$ and $\alpha \notin Z$. Obviously the characteristic function w_Z of the set Z (as a subset of L) is a classical valuation. In what follows, take the model $m \in \mathsf{M}_X$ such that w_Z is its classical valuation. From Lemma 3.3 it follows that $X \subseteq Z$, so for each $\beta \in X$, $1_m\ \beta$, due to Lemma 3.4, that is also for each $\beta \in Y$, $1_m\ \beta$. Obviously $w_Z(\alpha) = 0$. Then it has to be $7_m\ \alpha$, by the truth conditions for the formulae in the model m. Finally, $Y\ 7^{\mathsf{M}2}\ \alpha$.

Now consider the case: $\alpha \in \#X$ that is $\#\alpha \in X$. X is closed on (R2), therefore $\alpha \notin Var$ (since $\alpha \notin X$). So let α be of the form: $\beta \to \gamma$. In what follows, $\#(\beta \to \gamma) \in X$ and $\beta \to \gamma \notin X$ which, on the basis of the fundamental lemma for 0_2, yields $\beta > \gamma \notin X$. This means that in every model $m \in \mathsf{M}_X$, $7_m\ \beta \to \gamma$, by virtue of truth conditions. In other words, $\forall$m $\in \mathsf{M}_X$, $7_m\ \alpha$. In that way in order to complete the proof it is enough to find such a model $m \in \mathsf{M}_X$ that for each $\delta \in X$, $1_m\ \delta$. Just let m be a model from M_X determined by the classical valuation w which gives the truth value 1 to each variable $p \in Var$. Obviously, w is the characteristic function of the whole set of formulae L. Thus, taking Lemma 3.4 into consideration, $\forall\delta \in X$, $1_m\ \delta$.

As we have already noticed, the logic 0_2 is not structural: for example, $\{\#p\}\ 0_2\ p$ (comp. (R2)), however $\{\#(p \to q)\}\ 6_2\ p \to q$. Consider a weakening 0_2' of 0_2 determined by the set of rules for 0_2 without the only non-structural rule (R2). Obviously, 0_2' is a structural consequence relation, so the set of its theorems: $\{\alpha \in L\colon 0_2\text{'}\ \alpha\}$ is closed on every substitution of the language. It is easily seen that for any formula α, 0_2' α iff $1^{\mathsf{M}2}\ \alpha$ (the implication: $1^{\mathsf{M}2}\ p$ implies that 0_2' p, where $p \in Var$, holds since the antecedent is never true; while the implication: $1^{\mathsf{M}2}\ \beta \to \gamma$ implies that 0_2' $\beta \to \gamma$, can be proved in the same way as in the completeness proof for 0_2, obviously without using (R2), putting $Y = \varnothing$) which implies (via equality $1^{M_2} = 0_2$) that the set of theorems of 0_2 is closed on any substitution.

4. Analytic implications: classical versus strict

The results given in this section have been rather expected. For example, it seems to be obvious that the logics of analytic strict implication should be weaker than their classical counterparts. So first we can state:

Proposition 4.1. For any $\alpha \in L$, $1^{M_a}\ \alpha$ implies that $1^{M_1}\ \alpha$ and $\{\alpha \in L\colon\ 1^{M_a}\ \alpha\} \neq \{\alpha \in L\colon 1^{M_1}\ \alpha\}$.

Proof. The inclusion $1^{M_a} \subseteq 1^{M_1}$ is obvious due to the fact that $M_1 \subseteq M_a$ in the sense that each model $m \in M_1$ can be conceived as determined by non-essential for the definition of 1_m, additional parameter $\Xi = \{x\} \times \{x\}$, where x is the only point of m. In order to show that $\{\alpha \in L\colon\ 1^{M_a}\ \alpha\} \neq \{\alpha \in L\colon 1^{M_1}\ \alpha\}$ consider, for example, the axiom (Ax3) of the logic 1^{M_1} and prove that $7^{M_a}\ (p > q) \Rightarrow ((p \Rightarrow r) \Rightarrow (((p \rightarrow q) \Rightarrow r) \Rightarrow r))$, where p,q,r are different propositional variables. To this aim use Proposition 1.1 and show that $(p > q) \Rightarrow ((p \Rightarrow r) \Rightarrow (((p \rightarrow q) \Rightarrow r) \Rightarrow r)) \notin S4$. In order to do this, make use of the following three facts: for any formulas α, β, $\alpha > \beta \in S4$; in any model $<W,1_{S4}>$ for *S4* logic of strict implication, determined by a reflexive and transitive relation Ξ on W, for any $x \in W$, $x\ 1_{S4}\ \alpha \Rightarrow \beta$ iff $\forall y \in W$ (if $x\ \Xi\ y$, then either $\exists z \in W$ ($y\ \Xi\ z$ and $z\ 7_{S4}\ \alpha$) or $\forall z \in W$ ($z\ 1_{S4}\ \beta$ whenever $y\ \Xi\ z$)); and $x\ 1_{S4}\ \alpha \Rightarrow \beta$ implies that $\forall y \in W$ (if $x\ \Xi\ y$, then $y\ 1_{S4}\ \alpha \Rightarrow \beta$). Consider a model $<\{x,y,z\},1_{S4}\}>$, where the satisfiability relation 1_{S4} is determined by the ordering Ξ such that $x\ \Xi\ y\ \Xi\ z$ and the function $v\colon Var \longrightarrow P(\{x,y,z\})$ such that $y \in v(p)$, $y \notin v(q)$, $y \notin v(r)$, $z \notin v(p)$, $z \in v(r)$. One can show that $x\ 7_{S4}\ (p > q) \Rightarrow ((p \Rightarrow r) \Rightarrow (((p \rightarrow q) \Rightarrow r) \Rightarrow r))$.

As an obvious corollary of the proofs of Proposition 4.1 and 1.1(2) (notice that not only $7^{M_a}\ ((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow (p \rightarrow p))$ as it is established in the proof of Proposition 1.1 but also $7^{M_1}\ ((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow (p \rightarrow p))$ which can be proved exactly in the same way), incomparability of the sets of theorems of the second logic of analytic strict and the first logic of analytic classical implications can be established:

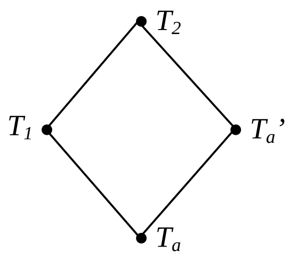
Proposition 4.2. Neither $\{\alpha \in L\text{: } 1^{M_1}\ \alpha\} \subseteq \{\alpha \in L\text{: } 1^{M_a}{}'\ \alpha\}$ nor $\{\alpha \in L\text{: } 1^{M_a}{}'\ \alpha\} \subseteq \{\alpha \in L\text{: } 1^{M_1}\ \alpha\}$.

Finally, we have a counterpart of Proposition 4.1, connecting the second logics of analytic strict and classical implication:

Proposition 4.3. For any formula α, $1^{M_a}{}'\ \alpha$ implies that $1^{M_2}\ \alpha$ and $\{\alpha \in L\text{: } 1^{M_a}{}'\ \alpha\} \neq \{\alpha \in L\text{: } 1^{M_2}\ \alpha\}$.

Proof. The fact that every theorem of the logic $1^{M_a}{}'$ is also a theorem of 1^{M_2} follows from the Propositions 1.1(1), 3.1(2) and obvious observation that $\mathsf{S4} \subseteq \{\alpha \in L\text{: } 1_{cl}\ \alpha\}$. The next fact that the sets of theorems of both logics are different can be illustrated, for example, by the formula: $((p \to q)\to(p \to q)) \to (p \to (q \to p))$ which is a theorem of 1^{M_2} and does not belong to $\mathsf{S4}$.

For a symbol $s \in \{1,2,a\}$, let $T_s = \{\alpha \in \mathrm{L}\text{: } 1^{M_s}\ \alpha\}$ and $T_a{}' = \{\alpha \in L\text{: } 1^{M_a}{}'\ \alpha\}$. Then the connections between the sets of theorems of logics of analytic implications considered in the paper can be formulated in the form of the following lattice:



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