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NEW AXIOMATIZATIONS OF $\mathbf{Ver}_n$

David B. MARTENS

Abstract
This note gives two new axiomatizations of each of the modal logics Ver_n by extension of K with, respectively, an inference rule and an axiom schema other than $\mathbf{Ver}_n$.

1. Introduction

Hughes and Cresswell describe the modal logics Ver_n as follows. "For any $n \geq 1$, let us call the system $\mathrm{K} + \Box^n p$, Ver_n. [E]ach Ver_n is characterized by the class of models in which everything that any world can see in $n - 1$ steps is a dead end." ([2], p. 36) For any $n \geq 1$, let us now call the axiom schema $\Box^n p$, which thus forms Ver_n from K, $\mathbf{Ver}_n$. This note gives two new axiomatizations of each Ver_n by extension of K with, respectively, an inference rule and an axiom schema other than $\mathbf{Ver}_n$.

2. First new axiomatization

Ver_n is axiomatizable as $\mathrm{K} + \mathrm{RVer}_n$, where RVer_n is RVer_1 when $n = 1$, and RVer_n is $\mathrm{RVer}_{n>1}$ when $n > 1$.

(RVer_1) If $\vdash \Box\alpha \supset \beta$ then $\vdash \beta$.
($\mathrm{RVer}_{n>1}$) If $\vdash \Box^{n-1}\alpha \supset \Box^{n-2}\beta$ then $\vdash \Box^{n-1}\beta$.

Lemma 2.1. $\mathbf{Ver}_1$ is provable in $\mathrm{K} + \mathrm{RVer}_1$.
Proof.

(1)	$\Box p \supset \Box p$	PC
(2)	$\Box p$	1, RVer_1

Lemma 2.2. $RVer_1$ is derivable in K + $\mathbf{Ver_1}$.
Proof.

1	(1)	$\vdash\Box\alpha\supset\beta$	premise
	(2)	$\vdash\Box\beta$	$\mathbf{Ver_1}$

Lemma 2.3. $\mathbf{Ver_{n>1}}$ is provable in K + $RVer_{n>1}$.
Proof.

(1)	$\Box^{n-1}p\supset\Box^{n-1}p$	PC
(2)	$\Box^{n}p$	1, $RVer_{n>1}$ [p/α, $\Box p/\beta$]

Lemma 2.4. $RVer_{n>1}$ is derivable in K + $\mathbf{Ver_{n>1}}$.
Proof.

1	(1)	$\vdash\Box^{n-1}\alpha\supset\Box^{n-2}\beta$	premise
1	(2)	$\vdash\Box(\Box^{n-1}\alpha\supset\Box^{n-2}\beta)$	1, N
	(3)	$\vdash\Box(\Box^{n-1}\alpha\supset\Box^{n-2}\beta)\supset(\Box^{n}\alpha\supset\Box^{n-1}\beta)$	**K**
1	(4)	$\vdash\Box^{n}\alpha\supset\Box^{n-1}\beta$	2, 3, MP
	(5)	$\vdash\Box^{n}\alpha$	$\mathbf{Ver_{n>1}}$
1	(6)	$\vdash\Box^{n-1}\beta$	4, 5, MP

3. Second new axiomatization

$RVer_n$ has an equivalent axiom form, Ver'_n. Thus, Ver_n is axiomatizable as K + Ver'_n, where Ver'_n is Ver'_1 when $n = 1$, and Ver'_n is $Ver'_{n>1}$ when $n > 1$.

(Ver'_1) $(\Box q\supset p)\supset p$
($Ver'_{n>1}$) $\Box(\Box^{n-1}q\supset\Box^{n-2}p)\supset\Box^{n-1}p$

Lemma 3.1. $\mathbf{Ver_1}$ is provable in K + Ver'_1.
Proof.

(1)	$(\Box q\supset p)\supset p$	Ver'_1
(2)	$(\Box p\supset\Box p)\supset\Box p$	1, US [p/q, $\Box p/p$]

	(3)	$\Box p$	2, PC

Lemma 3.2. Ver'_1 is provable in K + $\mathbf{Ver_1}$.
Proof.

	(1)	$\Box q$	$\mathbf{Ver_1}$
	(2)	$(\Box q \supset p) \supset p$	1, PC

Lemma 3.3. $\mathbf{Ver}_{n>1}$ is provable in K + $\mathrm{Ver}'_{n>1}$.
Proof.

	(1)	$\Box(\Box^{n-1}q \supset \Box^{n-2}p) \supset \Box^{n-1}p$	$\mathrm{Ver}'_{n>1}$
	(2)	$\Box(\Box^{n-1}p \supset \Box^{n-1}p) \supset \Box^{n}p$	1, US $[p/q, \Box p/p]$
	(3)	$\Box^{n-1}p \supset \Box^{n-1}p$	PC
	(4)	$\Box(\Box^{n-1}p \supset \Box^{n-1}p)$	3, N
	(5)	$\Box^{n}p$	2, 4, MP

Lemma 3.4. $\mathrm{Ver}'_{n>1}$ is provable in K + $\mathbf{Ver}_{n>1}$.
Proof.

	(1)	$\Box^{n}q$	$\mathbf{Ver}_{n>1}$
2	(2)	$\Box(\Box^{n-1}q \supset \Box^{n-2}p)$	assp. for CP
	(3)	$\Box(\Box^{n-1}q \supset \Box^{n-2}p) \supset (\Box^{n}q \supset \Box^{n-1}p)$	**K**
2	(4)	$\Box^{n}q \supset \Box^{n-1}p$	2, 3, MP
2	(5)	$\Box^{n-1}p$	1, 4, MP
	(6)	$\Box(\Box^{n-1}q \supset \Box^{n-2}p) \supset \Box^{n-1}p$	2–5, CP

4. Conclusion

Chellas and Segerberg mention a "general issue...of considerable interest: when is it possible to express a rule by an axiom schema? To the best of our knowledge there are only relatively few particular answers to this question and no general one." ([1], p. 23, n. 29) This note gives infinitely many further particular answers to the question.

Auburn University, dbm@ca.inter.net

REFERENCES

[1] Chellas, B.F., and Segerberg, K., "Modal Logics in the Vicinity of S1", *Notre Dame Journal of Formal Logic*, 37, 1996, pp. 1-24.
[2] Hughes, G.E., and Cresswell, M.J., *A Companion to Modal Logic*, Methuen, London and New York, 1984.