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FORMALIZATION OF INTENSIONAL FUNCTIONS AND EPISTEMIC KNOWLEDGE REPRESENTATION SYSTEMS*

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Abstract
Łoś (1948) formalization of intensional functions was made for the purpose of many-valued interpretation of the belief-operators within the scope of the classical logic system. The first aim of the paper is to present and discuss this rather unknown many-valued construction and its properties. The fact that the many-valuedness of Łoś systems is purely formal - their characteristic matrices are Boolean - calls for further consideration. Departing from intristic similarities of the tables for the epistemic operators to the information functions we show that Łoś structures may be rewritten as special knowledge representation systems. These systems use 0 and 1 as the only values and are called "epistemic". Their role for the theory of knowledge information systems may be compared to that of the functionally complete matrices in the class of all logical matrices for a given propositional language.

1. Belief functions and many-valuedness

Intensional propositional functions, in contrast to extensional functions, are those functions of propositional arguments whose logical value (its truth or falsity) does not depend solely on the logical values of the arguments. Therefore, the intensionality is identified with non-truth-functionality and many-valued logics may be considered as possible semantic base for their formalization.

Łoś in [1] focuses on the belief operators of the kind "John believes that *p*" or, more accurately, "John asserts that *p*", *p* being a proposition. He urged that the complete propositions of this form may be evaluated within the classical logic and, thus, under some reasonable assumptions, they may be formalized similarly as known modal connectives.

All propositions "John asserts that *p*" are clearly substitutions of the schema "*x* asserts that *p*", whose formal counterpart is a function *Lxp* of two arguments *x* and *p*, assigning a logical value to each couple (name, proposition). Łoś considers the language *L* including two kinds of variables, nominal and propositional, and, additionally, quantifiers. As axioms of the system for *L* he assumes the formulas:

(L1)	$Lxp \equiv \neg Lx(\neg p)$
(L2*i*)	$Lx(Ax_i)$, where (Ax_i) i {1,2,3} is the *i*-th axiom of (¬, →)-system of CPC of Łukasiewicz
(L3)	$Lx(p \to q) \to (Lxp \to Lxq)$
(L4)	$(\ x)\ Lxp \to p$
(L5)	$LxLxp \equiv Lxp$

accepting the rules: MP, the substitution rule extended onto the whole language *L* and the generalization rule. Note that e.g. (L3) expresses the fact that everyone uses the Detachment Rule i.e. asserting a conditional statement and acknowledging the antecedent of that statement commits one to asserting the consequent, or that (L4) says that a sentence acknowledged by everyone is a theorem of the system.

The operators *Lx*, *Ly*,... are certainly not the only intensional functions of the system considered. What is more, the closed formulas of the language define intensional propositional functions i.e. connectives; here the case is the function *S*:

$$S\alpha =_{df} \ x \ \ y\ (Lx\,\alpha \ \ \ Ly(\neg\alpha)),$$

which can be interpreted as "it is questionable that α". Intuitively, the definition conveys the thought that saying "it is debatable that α" means to say "two people exist such that one asserts α, and the other asserts not-α".

Any interpretation of the system of *L*-operators (and derived operators) starts with the selection of a definite range (domain) of nominal variables and a proposition set. The simplest is the case of two persons *A* and *B*; let us suppose that they do not agree in all the issues i.e. that in a certain definite set of propositions *Z* there are some accepted by *A* but not accepted by *B* and vice versa. Then the assent property divides *Z* into four classes, which shall be denoted symbolically as: 0, 1/3, 2/3, 1. The first class, 0, contains propositions which are acknowledged by neither person, the second class, 1/3, propositions which *A* acknowledges and *B* does not, the third class, 2/3, propositions which *B* acknowledges and *A* does not. Finally, the forth class, 1, all propositions acknowledged by *A* and *B* i.e. all logical theorems and perhaps other propositions acknowledged by both men. Identifying the acceptance of a proposition α by a person s with the formula $Ls\alpha$ we get a truth-table assigning to classes 0,1/3,2/3,1 the pairs of logical values of truth and falsity (notation t and f instead of 1 and 0 is to distinguish classical values from the values just introduced). In turn, making use of truth-tables for t and f we get the characterization of implication, negation and the connective S by means of the tables whose elements are the symbols of the four considered classes of propositions {0, 1/3, 2/3, 1} are identified with pairs (x,y), where x,y $\{f, t\}$, defined by the table for L. The connectives are defined just as in the product of matrices i.e. $\neg(x, y) = (\neg x, \neg y)$ and $(x_1, y_1) \to (x_2, y_2) = (x_1 \to x_2, y_1 \to y_2)$.

L	A	B
0	f	f
1/3	t	f
2/3	f	t
1	t	t

α	$\neg\alpha$	$S\alpha$
0	1	0
1/3	2/3	1
2/3	1/3	1
1	0	0

$\to$	0	1/3	2/3	1
0	1	1	1	1
1/3	2/3	1	2/3	1
2/3	1/3	1/3	1	1
1	0	1/3	2/3	1

Łoś suggests that the objects 0, 1/3, 2/3, 1 may be treated as logical values. He further considers the matrix

I4 = ({0,1/3,2/3,1}, ¬, →, {1})

demonstrating that E(I4) = TAUT{¬,→} (the set of classical (¬,→)-tautologies), and hence that many-valuedness of this system has but formal character.

The shifting of Łoś interpretation onto the cases with more persons (or, observers) is straightforward, and it results in next formally many-valued versions of CPC described by matrices with more than four elements. More precisely: for n observers (given a finite $n \geq 2$) one gets 2^n classes of propositions, which may be identified with n−tuples of t and f. Accordingly, we then define the connectives ¬ and → in the very same way as in the case considered above and what we get is a Boolean algebra described in terms of the two logical connectives, Distinguishing the biggest element 1 (the tuple of *t*'s), we finally get the 2^n element I2^n matrix for CPC.

2. Epistemic knowledge representation systems

It is obvious that any Łoś matrix is a finite Boolean algebra with 1 serving as the designated element. Leaving aside further philosophical issues which may be related with the Łoś construction we will discuss its compliance with the theory of knowledge representation systems, cf. [2], [4].

Recall that a knowledge representation system, *KRS* for short, is a triple

$$S = (U, A, V, \rho)$$

where $U = \{u_1, u_2, \ldots, u_n\}$ is a finite set of objects, $A = \{A_1, A_2, \ldots, A_k\}$ a finite set of attributes, and $V =$ V_{A_i} a set of values of attributes - V_{A_i} being the set of values for a given attribute A_i. Finally, ρ is a total function from the set $U \times A$ into V, $\rho: U \times A \to V$, such that for any u U and A_i A, $\rho(u,A_i)$ V_{A_i}. It would be in order to add that in general there no limitations on the nature of "objects"' are imposed and in particular they may be states, processes etc. The "attributes" are properties of objects and they resemble logical predicates with this difference that contrary to predicates which correspond to properties which hold or do not hold, the attributes may take as their values scales, numbers and other values.

Given *S* as above, the relation ≈ on *U* defined by the formula

$$x \approx y \text{ if and only if } \rho(A,x) = \rho(A,y) \text{ for every attribute } A$$

is an equivalence relation. If all equivalence classes are one-element sets the *KRS* is called *selective*, cf. [3].

It is straightforward that any 2^n element Łoś table for *L* defines a knowledge representation system and the only difference is that now *A*, *B*,… should be treated as attributes and not observers while the non-classical logical values, e.g. 0, 1/3, 2/3, 1 as objects. Obviously then *t* and *f* would play their usual roles of the classical logical values. Thus, in the case considered in Section 1 we get the *KRS* with $U = \{0, 1/3, 2/3, 1\}$, $A = \{A, B\}$ and $V = \{f, t\}$, the table for *L* being now the table for ρ.

Generally, by a Łoś *knowledge representation system*, *LKRS* for short, we shall understand the system composed of

(i) 2^n element set of objects $U = \{u_1, u_2, \ldots, u_{2^n}\}$, $n \geq 2$, n finite;

(ii) the set of n attributes $\{A_1, A_2, \ldots, A_n\}$

(iii) the set of two values $V = \{0, 1\}$, and

(iv) the function ρ such that $\{(\rho(u_i, A_j) : j \quad \{1,\ldots,n\}) : 1 \leq i \leq 2^n\}$ is the set of all mutually different n-tuples of 0's and 1's, i.e. the system has the maximal possible numbers of mutually independent attributes. In general its table would look like the following one:

ρ	A_1	A_2	A_3	...	A_{n-2}	A_{n-1}	A_n
u_1	0	0	0	...	0	0	0
u_2	1	0	0	...	0	0	0
u_3	1	1	0	...	0	0	0
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
u_t	0	1	0	...	1	0	1
u_{t+1}	1	0	1	...	0	1	0
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$u_{2^{n-2}}$	0	0	1	...	1	1	1
$u_{2^{n-1}}$	0	1	1	...	1	1	1
u_{2^n}	1	1	1	...	1	1	1

When the set of values V in a *KRS* consists of two elements only, say $V=\{0,1\}$, the device becomes close to the classical logic since the attributes are 0-1 -valued predicates. In the sequel we shall refer to any

$S = (U, A, \{0, 1\}, \rho)$

as *epistemic*.

Remark. For a given finite n ≥ 2, the *LKRS* is a selective epistemic knowledge representation system.

It is rather straightforward that to any knowledge representation system its epistemic counterpart in a unique way may be associated. All sets are finite and the set of attributes *A* may be replaced with *A*' consisting of *compositional* predicates of A_i's and their possible values. Clearly, then the resulting system will have much more attributes ($k \cdot l$, where *l* is the cardinal of *V*). However, taking into account its simplicity and "logical" character it may be much convenient to operate with.

Example. Let *S* be the following knowledge representation system:

$U = \{u_1, u_2, u_3, u_4\}$,
$A = \{A, B\}$,
$V = V_A \quad V_B$ with $V_A = \{a_1, a_2\}$, $V_B = \{b_1, b_2\}$,

and let ρ be defined through the table:

ρ	*A*	*B*
u_1	a_1	b_1
u_2	a_1	b_2
u_3	a_2	b_1
u_4	a_2	b_2

Retaining the set of objects *U* and considering the set of attributes $A' = \{A_1, A_2, B_1, B_2\}$ which are "compound" predicates corresponding to $A(a_1)$, $A(\mathrm{a}_2)$, $B(b_1)$ and $B(b_2)$, respectively, we get the epistemic system *S'* equivalent to *S* assuming that ρ' is given by the table:

ρ'	A_1	A_2	B_1	B_2
u_1	1	0	1	0
u_2	1	0	0	1
u_3	0	1	1	0
u_4	0	1	0	1

Notice that the both till considered systems *S* and *S'* are *selective*, i.e. they discern all the objects involved, see [3]. On the other hand, extending the range of objects one may arrive at the selective system having 16 objects $w_1, w_2, \ldots, w_{16}$ and ρ" characterized by the table:

ρ”	A_1	A_2	A_3	A_4
w_1	0	0	0	0
w_2	1	0	0	0
w_3	1	1	0	0
w_4	1	1	1	0
w_5	0	1	1	0
w_6	0	1	1	1
w_7	0	1	0	1
w_8	0	0	1	0
w_9	1	1	0	1
w_{10}	1	0	1	0
w_{11}	1	0	0	0
w_{12}	1	0	0	1
w_{13}	0	0	0	1
w_{14}	0	0	1	1
w_{15}	0	1	1	1
w_{16}	1	1	1	1

The last system, say *S”* is the unique *LKRS* for $n = 4$.

Notice that *S”* is the biggest possible selective epistemic knowledge representation system with four attributes and consequently, that it “contains all other such systems of four attibutes”. Thus, e.g. *S’* considered above is a subsystem of *S”* based on $\{w_{10}, w_{12}, w_5, w_7\}$, where $B_1 = A_3$ and $B_2 = A_4$. Finally, since any such reduction may be considered in terms of valuation functions, the Łoś knowledge representation systems play similar role in the class of KRS as functionally complete systems in the theory of logical matrices, cf. [5].

3. Prospects

Epistemic knowledge representation systems are logically “transparent”. The logical structure of any such system is displayed by the table defining the information function ρ and the underlying “information logic” operates on simple descriptors of the form (*A*,0) and (*A*,1) and thus, practically, on one-argument predicates. Special epistemic knowledge representation systems, which are object-reducts of Łoś knowledge representation systems, are selective. In such systems, any information (i.e. a subset of *U*) is definable directly from the tables for information function using Boolean connectives.

Though the reduction of an arbitrary *KRS* to the epistemic system increases its complexity, i.e. the number of attributes, the rewarding feature of considering epistemic systems is a possibility of using techniques of minimalization and reduction known from the switching circuits theory. Another advantage is the structural simplicity of the decision tables for several concrete problems e.g. for program specification in software engineering.

It is straightforward that direct representing knowledge representation systems in terms of their epistemic counterparts does not change their essential features. Thus, if a given system *S* has some mutually dependent attributes, its epistemic version *S'* will also have such attributes. One may then ask on which stage of consideration one should reduce the dependent attributes. Namely, whether in the original system *S* or in its epistemic counterpart *S'*. It seems that the answer to this and other similar questions would depend on the peculiarity of *S* and of the kind of particular problem to solve.

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REFERENCES

[1] Łoś, J., 'Logiki wielowartościowe a formalizacja funkcji intensjonalnych' (Many-valued logics and the formalization of intensional functions), *Kwartalnik Filozoficzny*, **17**, 1948, pp. 59-78.

[2] Orłowska, E. and Pawlak, Z., 'Logical foundations of knowledge representation', *Institute of Computer Science, Polish Academy of Sciences Reports*, **537**,1984a, pp. 1-106.

[3] Orłowska, E. and Pawlak, Z (1984b). *Expressible power of knowledge representation systems*. Int. J. of Man-Machine Studies, 20, pp. 485-500.

[4] Pawlak, Z., 'Rough sets', *Theoretical aspects of reasoning about data*, Kluwer Academic Publishers, 1991.

[5] Wójcicki, R., *Theory of logical calculi. Basic theory of consequence operations*. Kluwer Academic Publishers, Dordrecht, 1988.