Logica Trianguli, 1, 1997, 59-71

ON MANY-VALUEDNESS, SENTENTIAL IDENTITY, INFERENCE AND ŁUKASIEWICZ MODALITIES

Grzegorz MALINOWSKI

Abstract

The development of the method of logical matrices at the turn of 19th Century made it possible to define the concept of many-valued logic. Since the first construction of the system of three-valued logic by Łukasiewicz in 1918 several matrix based logics have been proposed, cf. [8]. The aim of the present paper is to touch upon some problems related to the topic, which would permit one to get a viewpoint upon the nature of many-valuedness.

First, we show that the multiplication of logical values is not a sufficient condition to obtain a non-two-valued logic. Second, we discuss an ingenious solution by R. Suszko [11] explaining through the sentential identity an ontological nature of non-classical logical values. Next, we present a kind of metalogical relation of inference, so-called q-consequence, being three-valued in its spirit. The last chapter will bring a concise description of two Łukasiewicz "many-valued" systems of modalities and an application of the paradigm of q-consequence to these systems.

1. Many-valuedness

A generic construction of a many-valued logic starts with the choice of the sentential language *L* which may be shown as an algebra $L = (For, F_1, \ldots, F_m)$ freely generated by the set of sentential variables $Var = \{p, q, r, \ldots\}$. Formulas, i.e. elements of *For*, are then built from variables using the operations $F_1, \ldots, F_m$ representing the sentential connectives. In most cases, either the language of the classical sentential logic:

$$L_k = (For, \neg, \rightarrow, \quad, \quad, \leftrightarrow)$$

with negation (¬), implication (→), disjunction ( ), conjunction ( ), and equivalence (↔), or some of its reducts is considered. Subsequently, one defines a multiple-valued algebra *A* similar to *L* and chooses a nonempty subset of the universe of A, D A, of *designated* (or *distinguished*) elements. The interpretation structures

$$M = (A\ ,\ D)$$

are called *logical matrices*.

Given a matrix M for a language L, the *system* E(M) of sentential logic is defined as the *content* of M, i.e. the set of all formulas which take designated values for every valuation h (a homomorphism) of L in M. Thus

$$\mathrm{E}(M) = \{\ \alpha \quad For : \text{for every } h \quad \mathrm{Hom}(L,A),\ h(\alpha) \quad D\ \}.$$

The notion of matrix consequence being a natural generalisation of the classical consequence is defined as follows: relation $\models_M$ is said to be a matrix consequence of M provided that for any X For, α For

$X \models_M \alpha$ if and only if for every h Hom(L,A) ($h(\alpha)$ D whenever $h(X)$ D).

The example given below shows that the choice of multiple-valued algebra as a base for either of the logical paradigm does not guarantee the many-valuedness of the construction:

1.1. Consider the matrix

$$M_3 = (\ \{0,t,1\}\ ,\ \neg\ ,\ \rightarrow\ ,\quad ,\quad ,\ \leftrightarrow\ ,\ \{t,1\}\ )$$

for L_k with the operations defined by the following tables:

x	¬x
0	1
t	0
1	0

→	0	t	1
0	1	t	1
t	0	t	t
1	0	t	1

	0	t	1
0	0	t	1
t	t	t	1
1	1	1	1

	0	t	1
0	0	0	0
t	0	t	t
1	0	t	1

↔	0	t	1
0	1	0	0
t	0	t	t
1	0	t	1

We claim that this three-valued matrix determine both the system of tautologies of the classical logic *TAUT* and the classical consequence relation $\models_2$. Recall that *TAUT* = E(M_2) and $\models_2 = \models_{M_2}$, where

M_2 = ({0,1} , ¬ , → ,   ,   , ↔ , {1})

and the connectives are defined by the classical truth-tables. To verify that it suffices to notice that due to the choice of the set of designated elements {t,1}, with each *h*   *Hom*(*L*,*A*) the valuation *h**   *Hom*(*L*, M_2) corresponds in a one-to-one way such that *h*(α)   {t,1} iff *h*(*α) = 1. Thus, the logic under consideration in neither sense is many-valued.

The second example of the three-element matrix logic is even more surprising. There we are given the matrix determining as its system the same set of classical tautologies, but its consequence relation is non-classical.

1.2. Consider the matrix

M_3 = ({0,t,1} , ¬ , → ,   ,   , ↔ , {t,1})

for L_k with the operations defined by the following tables:

x	¬ x
0	1
t	1
1	0

→	0	t	1
0	1	1	1
t	1	1	1
1	0	0	1

	0	t	1
0	0	0	1
t	0	0	1
1	1	1	1

	0	t	1
0	0	0	0
t	0	0	0
1	0	0	1

$\leftrightarrow$	0	t	1
0	1	1	0
t	1	1	0
1	0	0	1

Taking into account that t and 0 are indistinguishable by the truth tables in formulas containing the connectives, thus practically in all formulas except the propositional variables, and that both values are designated, we obtain $E(M_3) = TAUT$. Accordingly, M_2 is the only two-element matrix which might determine $\models_{M_3}$. Simultaneously, $\models_{M_3} \neq \models_{M_2}$, since, for example,

$$\{p \rightarrow q, p\} \models_{M_2} q \quad \text{while not} \quad \{p \rightarrow q, p\} \models_{M_3} q.$$

To verify this it simply suffices to turn over a valuation h such that hp = t and $hq = 0$

2. Sentential identity

A very special property of the behaviour of the classical equivalence connective may be used to express the fact that the only attribute of sentence which counts for the classical logic is its truth-value. This is due to the following truth table condition for the function of the equivalence:

$$x \leftrightarrow y \quad \{1\} \quad \text{if and only if} \quad x = y$$[1].

Let us note that the equality appearing on the right hand side of the formula is merely the *identity of the logical values* and not identity of sentences in any extended or deeper sense. All that is in accordance with the Fregean condition stating that, from the point of view of the (classical) logic, two sentences having the same logical values, describe the same, i.e. have the same *referent* or, denotation, cf. [2]. Accordingly, since the truth tables cover only a small part of the ontology of

[1] $x \leftrightarrow y \quad \{1\}$ means, obviously, that $x \leftrightarrow y = 1$.

referents of sentences and in no reasonable sense they may tell anything about the contents of these linguistic entities.

For the purpose of avoiding this inconvenience, R. Suszko [11] extended the classical logic introducing in the language of the classical logic a non truth-functional connective of identity, denoted henceforth as ≡. The intended meaning of the new connective is the best explained through models, i.e. matrices and it consists in expressing the fact that two sentences are identical, *modulo* a given model, whenever their semantic correlates are identical. Relatively to a choice of the class of models one gets different kinds of sentential identity, which applies to diverse structures of universes semantic correlates including the distinctions between distinguished *situations*, i.e. those which obtain, and not distinguished, or negative. The weakest logic of sentential identity SCI, the Sentential Calculus of Identity, may be characterised semantically by the use of SCI-models, cf. [1]. Actually, an SCI-model is a (proper) matrix *M* = (*A* , *D*), consisting of an algebra

$$A = (\mathrm{A} , \neg, \rightarrow, \quad , \quad , \leftrightarrow, \equiv)$$

such that for any a, b A

¬ a *D*	if and only if	a *D*
a → b *D*	if and only if	a *D* and b *D*
a b *D*	if and only if	a *D* and b *D*
a b *D*	if and only if	a *D* and b *D*
a ↔ b *D*	if and only if	either a, b *D* or a, b *D*
a ≡ b *D*	if and only if	a = b.

The *referentially* defined SCI consequence relation $\models_{\mathrm{SCI}}$ is introduced as follows:

$$X \models_{\mathrm{SCI}} \alpha \text{ if and only if } X \models_{M} \alpha \text{ for every SCI-model } M.$$

SCI admits a great divergence of models and there are no limitations on either cardinality or the internal algebraic structure of an intended model, cf. [9]. Since, however, each interpretation of the SCI language *L* = (*For*, ¬, →, , , ↔, ≡) is a homomorphism *h* of *L* into *A* we may easily associate a bivalent logical valuation $t_h : For \rightarrow \{0,1\}$ so that

$t_h(\alpha) = 1$ if and only if $h(\alpha) \quad D$.

Then, obviously, t_h in each case is a usual valuation of the truth-functional connectives as described by the classical truth tables. As for the identity, we have

$t_h(\alpha \equiv \beta) = 1$ if and only if $h(\alpha) = h(\beta)$,

what is a translation of the last condition defining the SCI-model. On the other hand, using other conditions it is easy to verify that the truth-functional connectives of L behave in the same way with respect to t_h as they did with respect to usual {0,1}-valuation, i.e.

$t_h(\neg\alpha) = 1$	if and only if	$t_h(\alpha) = 0$
$t_h(\alpha \rightarrow \beta)) = 0$	if and only if	$t_h(\alpha) = 1$ and $t_h(\beta)) = 0$
$t_h(\alpha \quad \beta)) = 0$	if and only if	$t_h(\alpha) = 0$ and $t_h(\beta)) = 0$
$t_h(\alpha \quad \beta)) = 1$	if and only if	$t_h(\alpha) = 1$ and $t_h(\beta)) = 1$
$t_h(\alpha \leftrightarrow \beta)) = 1$	if and only if	$t_h(\alpha) = t_h(\beta))$.

This shows how referential assignments are related to logical valuations and, thus, how logical *two-valuedness* is opposed to referential *many-valuedness*.

2.1. (cf. [12]). Let us consider the three-element matrix M = (A3 , {1}) based on the algebra

A3 = ($\{0, {}^1/_2, 1\}$, ¬, →, , , ↔, ≡)

with the operations defined by the following tables:

x	¬ x
0	1
$^1/_2$	1
1	0

→	0	$^1/_2$	1
0	1	1	1
$^1/_2$	1	1	1
1	0	$^1/_2$	1

	0	$^1/_2$	1
0	0	$^1/_2$	1
$^1/_2$	$^1/_2$	$^1/_2$	1
1	1	1	1

	0	$^1/_2$	1
0	0	0	0
$^1/_2$	0	$^1/_2$	$^1/_2$
1	0	$^1/_2$	1

↔	0	$^1/_2$	1
0	1	1	0
$^1/_2$	1	1	$^1/_2$
1	0	$^1/_2$	1

≡	0	$^1/_2$	1
0	1	$^1/_2$	0
$^1/_2$	$^1/_2$	1	$^1/_2$
1	0	$^1/_2$	1

A straightforward verification proves that *M* is an SCI-model. There ¬, →, , , ↔ are classical connectives of negation, implication, disjunction, conjunction and equivalence, and ≡ the identity connective.

Now, let us define further sentential connectives , putting:

x $=_{df}$ (x ≡ ¬ (x ≡ x)) and
x y $=_{df}$ ((x y) ≡ x) or,
x y $=_{df}$ ((x y) ≡ y).

The tables of these connectives are then the following:

x	x
0	1
$^1/_2$	$^1/_2$
1	0

	0	$^1/_2$	1
0	1	1	1
$^1/_2$	$^1/_2$	1	1
1	0	$^1/_2$	1

i.e. and are the connectives of negation and implication of Łukasiewicz. Further to this Ł3 = ({0, $^1/_2$,1} , , , , , ≡ , {1}) is the matrix of the three-valued logic of Łukasiewicz, cf. [3].

3. Inference

In [10] a generalisation of Tarski's concept of consequence operation related upon the idea that the rejection and acceptance need not be complementary was proposed. The central notions of the framework are counterparts of the concepts of matrix and consequence relation - both distinguished by the prefix "*q*" which may be read as "quasi".

Where *L* is a sentential language and *A* is an algebra similar to *L*, a *q-matrix* is a triple

$$M^* = (A , D^* , D),$$

where D^* and D are disjoint subsets of the universe A of *A*, $D^* \cap D =$. D^* are then interpreted as sets of *rejected* and *distinguished* elements values of M^*, respectively. For any such M^* one defines the relation $\vdash_{M^*}$ between sets of formulae and formulae, a *matrix q-consequence of* M^* putting for any X For, α For

$X \vdash_{M^*} \alpha$ if and only if for every h Hom(L,A) ($h(\alpha)$ D whenever $h(X) \cap D^* =$).

The relation of q-consequence was designed as a formal counterpart of reasoning admitting rules of inference which from non-rejected assumptions lead to accepted conclusions. The q-concepts coincide with usual concepts of matrix and consequence only if D^* D = A, i.e. when the sets D^* and D are complementary. Then, the set of rejected elements coincides with the set of non-designated elements.

For every h Hom(L,A) let us define a three-valued function k_h : $For \rightarrow \{ 0 , {}^1/_2 , 1 \}$ putting

$$k_h(\alpha) = \begin{cases} 0 & \text{if} \quad h(\alpha) \quad D^* \\ {}^1/_2 & \text{if} \quad h(\alpha) \quad A - (D^* \quad D) \\ 1 & \text{if} \quad h(\alpha) \quad D. \end{cases}$$

Given a q-matrix M^* for L let $KV_M = \{k_h : h$ Hom(L,A)$\}$; we get the following three-valued description of the q-consequence relation $\vdash_{M^*}$:

$X \vdash_{M^*} \alpha$ if and only if for every k_h KV_M ($k_h(X) \cap \{0\} =$ implies $k_h(\alpha) = 1$).

It is worth emphasising that this description in general is not reducible to the two-valued description possible for the ordinary (structural) consequence relation. As the latter property may be interpreted as logical two-valuedness of logics identified with the consequence, we may say that a q-logic is logically either two or three valued. Moreover, the three-valued q-logics exist, cf. [7]. The example below shows that it is the case.

3.1. Consider the three-element q-matrix

$$Ł_q3 = (\{0, {}^1/_2, 1\} , \quad , \quad , \quad , \quad , \equiv , \{0\} , \{1\}),$$

where the connectives are defined as in the Łukasiewicz three-valued logic. Then, for any p *Var,* it is not true that $\{p\} \vdash_{M^*} p$. To see this, it suffice to consider the valuation sending p into $^1/_2$.

The more striking is perhaps the fact that even logics generated by some two-element q-matrices are three-valued. This is illustrated by our last example:

3.2. Let us consider the two-element algebra

$$A_2 = (\{0 , 1\}, \neg , \rightarrow , \quad , \quad , \leftrightarrow),$$

with the operations defined by the classical truth-tables of negation, implication, disjunction and equivalence. Next, let us consider the following two q-matrices:

$$M_1 = (A_2 , \quad , \{1\}),$$
$$M_0 = (A_2 , \quad , \{0\}).$$

The q-consequence relations of M_1 and M_0 are such that for any X *For*, α *For*

$$X \vdash_{M1} \alpha \text{ if and only if for every } h \quad \mathrm{Hom}(L,A_2)\ h(\alpha) = 1,$$

$$X \vdash_{M0} \alpha \text{ if and only if for every } h \quad \mathrm{Hom}(L,A_2)\ h(\alpha) = 0.$$

Thus, in the first case a formula α is a q-consequence of any set of formulas, whenever it is a tautology. In the second case α is a contradictory formula.

The standard description of $\vdash_{M1}$ in terms of $\{0, {}^1/_2, 1\}$-valuations k_h is then defined in such a way that for every α *For,* $k_h(\alpha) = 1$ iff α *TAUT,* $k_h(\alpha) = {}^1/_2$ otherwise; for no formula k_h takes the value 0.

Similarly, $X \vdash_{M0} \alpha$ whenever $k_h(\alpha) = 0$, where $k_h(\alpha) = 0$ iff α is contradictory and $k_h(\alpha) = {}^1/_2$ otherwise.

4. Łukasiewicz modalities

What appears to be one of the main Łukasiewicz's intentions in the course of construction of the three-valued logic is working out a tool to formalise the non-truth-functional functors of possibility (M) and necessity (L). Adopting the established relation of mutual definability

$$L\alpha = \quad M \quad \alpha$$

he put forward a minimal postulate to preserve in the logic acquired the consistency of everything inherited from the earlier intuitive theorems on modal propositions. The definition of possibility connective satisfying these requirements given by Tarski in 1921,

$$M\alpha = \quad \alpha \quad\quad \alpha,$$

led to the following tables of M and L:

x	Mx	Lx
0	1	1
$^1/_2$	1	0
1	0	0

In spite of the promising combination of trivalence and modality the full elaboration of modal logic on the basis of the three-valued logic never succeeded, which was the result of Łukasiewicz's further investigations, cf. [4], on modal sentences and finally resulted in the construction of another, four-valued, system of modal logic in [5].

The algebra of logical values of Łukasiewicz system of four-valued propositional logic Ł was a product of two Boolean algebras with implication, negation and one-argument operations of: *assertion* A (the first) and *verum* V (the second); i.e., ({0,1}, →, ¬, A) and ({0,1}, →, ¬, V), where A(0) = 0, A(1) = 1, and V(0) = V(1) = 1. The values were, primarily, the ordered pairs (1,1), (1,0), (0,1), (0,0). The product had three operations: → (implication), ¬ (negation) and Δ (possibility), identified with the "cross" product of A and V. Łukasiewicz also considered the second "possibility" twin to Δ. He also simplified the notation and used 1 to stand for (1,1), 2 for (1,0), 3 for (0,1) and 4 for (0,0). The Łukasiewicz logic algebra in this notation has the form:

Ł = ({ 1, 2, 3, 4 } , ¬ , → , Δ ,),

with operations defined by the following tables:

→	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

	¬
1	4
2	3
3	2
4	1

	Δ
1	1
2	1
3	3
4	3

1	1
2	2
3	1
4	2

The system Ł of modal logic was defined on the language *L* = (For , ¬ , → , Δ ,) as the set of all formulas taking for every valuation *h* (i.e., a homomorphism) of *L* in Ł the *distinguished value* 1, thus

Ł = { α *For* : for every *h* Hom(*L*, Ł), *h*(α) = 1 }.

The very special property of the two modal connectives (of possibility), already known and stressed by Łukasiewicz is that they are indistinguishable one from another, cf. [5]

First, let us consider the three-element q-matrix

$Ł_{M,L}$ = ({0, $^1/_2$,1} , , , , , ≡ , M , L , {0} , {1})

being the definitional extension of the q-matrix for the three-valued Łukasiewicz logic described in 3.1. Let now $\vdash_{M^*}$ denote the q-consequence relation defined by $Ł_{M,L}$ on the language containing, besides the Łukasiewicz usual connectives, also M and L. One may easily check that

$\alpha \vdash_{M^*} M\alpha$ and $L\alpha \vdash_{M^*} \alpha$,

and

neither $\alpha \vdash_{M^*} L\alpha$ nor $M\alpha \vdash_{M^*} \alpha$.

The first two inferences correspond to the following tautologies of the extended three-valued logic of Łukasiewicz:

$$\alpha \quad M\alpha \text{ and } L\alpha \quad \alpha,$$

the two latter are not universally valid formulas of this logic.

Now let us consider the following two q-matrices related to the Łukasiewicz system Ł:

$$\mathit{MŁ}_{\Delta} = (£ , \{ 3, 4\} , \{ 1\}),$$

$$\mathit{MŁ} \quad = (£ , \{ 2, 4\} , \{ 1\}).$$

The choice of the sets of rejected and accepted elements in $\mathit{MŁ}_{\Delta}$ and in $\mathit{MŁ}$ and the whole idea of considering inferential extensions of the system of modal logic are in a way connected with Łukasiewicz attempts to discern the two operators of possibility. Note, that in the first case rejected are those elements of the algebra of values which Δ “sends to” not designated values, i.e., different from 1.

The q-matrices $\mathit{MŁ}_{\Delta}$ and $\mathit{MŁ}$ define two q-consequence relations $\vdash_{\Delta}$ and $\vdash$. Since

$$\{ \alpha : \quad \vdash_{\Delta} \alpha \} = \{ \alpha : \quad \vdash \quad \alpha \} = Ł$$

the two logics are both (different) inferential extensions of the original system Ł of modal logic, cf. [6]. Moreover, the two permit in a natural way to make a distinction between the two “indistinguishable” possibilities. Namely,

$$\alpha \vdash_{\Delta} \Delta\, \alpha \text{ but } \textit{not } \alpha \vdash_{\Delta} \quad \alpha$$

and

$$\alpha \vdash \quad \alpha \text{ but } \textit{not } \alpha \vdash \quad \Delta\, \alpha.$$

Comment. Łukasiewicz modalities in the three-valued logic have been distinguished using a single q-consequence and the modalities of Ł modal system using two such inference relations. This, we claim, corresponds to the very fact that in one case the logic was based on the three-element matrix and in the other on the four-element matrix. Note, that the q-consequence relations defined by the four-element matrices related to the Łukasiewicz modal system, $MŁ_{\Delta}$ and *MŁ* , are *logically* three-valued.

University of Łódź, gregmal@krysia.uni.lodz.pl

REFERENCES

[1] Bloom, S., "A completeness theorem for 'Theories of kind W' ", *Studia Logica*, XXVII, 1971, 43-55.

[2] Frege, G., "Über Sinn und Bedeutung", *Zeitschrift für Philosophie und philosophische Kritik C*, 1892, 25-50.

[3] Łukasiewicz, j., "O logice trójwartościowej", *Ruch Filozoficzny*, 5, 170-171. English tr. "On three-valued logic" [in:] Borkowski, L. (ed.) *Selected works*, North-Holland, Amsterdam, 87-88.

[4] Łukasiewicz, J., "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie Cl. III*, 23, 51-77.

[5] Łukasiewicz, J., "A system of modal logic", *The Journal of Computing Systems*, 1, 1953, pp. 111-149.

[6] Malinowski, G., "Inferential extensions of Łukasiewicz modal logic", *an invited lecture to the Conference "Łukasiewicz in Dublin"*, University College Dublin, Department of Philosophy, Dublin 7 - 10 July 1996.

[7] Malinowski, G., "Inferential many-valuedness" [in:] Woleński, J. (ed.) *Philosophical logic in Poland*, Synthese Library, 228, Kluwer Academic Publishers, Dordrecht, 1994, 75-84.

[8] Malinowski, G., *Many-valued logics*, Oxford Logic Guides, 25, 1993, Clarendon Press, Oxford.

[9] Malinowski, G., "Notes on Sentential Logic with Identity", *Logique et Analyse*, 112, 1985, 341-352.

[10] Malinowski, G., "Q-consequence operation", *Reports on Mathematical Logic*, 24, 1990, 49-59.

[11] Suszko, R., "Abolition of the Fregean Axiom" [in:] Parikh, R. (ed.) *Logic Colloquium, Symposium on Logic held at Boston, 1972-73. Lecture Notes in Mathematics*, vol. 453, 1972, 169-239.

[12] Suszko, R., "Remarks on Łukasiewicz's three-valued logic", *Bulletin of the Section of Logic*, vol. 4, no. 3, 1975, pp. 87-90.