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THE LAW OF EXCLUDED MIDDLE AND INTUITIONISTIC LOGIC

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Abstract

This paper is a proposal of continuation of the work of C. Rauszer. The logic of falsehood created by her may constitute the starting point for construction of logic formalising reductive reasonings. The extension of Heyting-Brouwer logic (HB) to its deductive-reductive form sheds new light upon those classical tautologies which are rejected in intuitionism. It turns out that among HB-tautologies there can be found all the classical ones. Some of them are characteristic for deductive reasoning and they are accepted by intuitionism. Others formulate the laws of reductive reasoning. Many of them, including the law of excluded middle has been rejected in Heyting's intuitionism. Intuitionism only permits for those reductive tautologies, which at the same time bear deductive character.

Thus, the complete HB intuitionism does not reject any of the classical tautologies. Every classical tautology appears in HB logic in its deductive or reductive part. Some are even present in both parts.

At the end it is shown that the law of excluded middle $\alpha \quad \neg\alpha$ is a law of the reductive part of the traditional, Heyting's intuitionistic propositional calculus.

1. HB logic and its logic of falsehood

The aim of this paper is to continue the work started and developed by Rauszer [4]-[8] and devoted to the so-called HB logic (Heyting-Brouwer logic).

In the 70-ies Rauszer constructed propositional calculus being an extension of traditional intuitionism (here called Heyting's) with a new part indefinable on the basis of implication called Brouwerian intuitionism. Referring to [7] let us remind that the language of the classical propositional calculus has been extended with two Brouwerian connectives: coimplication $\leftharpoondown$ and weak negation .

$$L_{HB} = (L_{HB}, \neg, \quad, \quad, \quad, \rightarrow, \leftharpoondown)$$

On the basis of so determined HB-language the axiomatization of HB logic is formed by axioms of Heyting's intuitionism and formulas:

$\alpha \to (\beta \quad (\alpha \leftarrow \beta))$
$(\alpha \leftarrow \beta) \to \quad (\alpha \to \beta)$
$((\alpha \leftarrow \beta) \leftarrow \gamma) \to (\alpha \leftarrow (\beta \quad \gamma))$
$\neg(\alpha \leftarrow \beta) \to (\alpha \to \beta)$
$(\alpha \to (\beta \leftarrow \beta)) \to \neg\alpha$
$\neg\alpha \to (\alpha \to (\beta \leftarrow \beta))$
$((\beta \to \beta) \leftarrow \alpha) \to \quad \alpha$
$\quad \alpha \to ((\beta \to \beta) \leftarrow \alpha)$

for $\alpha,\beta,\gamma \quad L_{HB}$. HB has two rules of inference: Modus Ponens and the rule: $\alpha \vdash \neg \quad \alpha$. Symmetry between implication and coimplication is confirmed by algebraic interpretation recalled by Rauszer. If a,b,x are elements of any Heyting-Brouwer algebra[1], then

$$x \leq a \to b \quad \text{iff} \quad a \cap x \leq b \qquad \text{and} \qquad a \leftarrow b \leq x \quad \text{iff} \quad a \leq b \quad x.$$

Similarly, for both negations

$$\neg a = a \to 0 \qquad \text{and} \qquad a = 1 \leftarrow a$$

with $1=(a \to a)$ and $0=(a \leftarrow a)$. Moreover

$$a \to b = 1 \qquad \text{iff} \qquad a \leq b \qquad \text{iff} \qquad a \leftarrow b = 0.$$

Of course, discussing a subject of algebraic operation $\leftarrow$ it is impossible to omit already historical, very important [1], Mc Kinsey and Tarski paper.

Upon the same HB-language, Rauszer builds a Heyting-Brouwer logic of falsehood (named $\overline{HB}$) as a mirror picture of HB. Here coimplication takes such a role as implication in Heyting's intuitionism. It can be interpreted as a metalogic negation of implication. Therefore the main function traditionally belonging to Modus Ponens is taken over by Modus Tollens: $\alpha \leftarrow \beta, \beta \vdash \alpha$. The other rule of inference is $\quad \neg\alpha \vdash \alpha$. As it can be easily noticed, $\overline{HB}$ preserves falsity. The fact that $\alpha \leftarrow \beta$ is false

[1] For details concerning with a Heyting-Brouwer algebra see [4] or [7].

means that β follows from α. Thus, if we suppose that β is false, then by deduction we have falsehood of α.

The set of axioms for $\overline{HB}$ presented in [7] consists of the following formulas:

(E$_1$) $((\beta \leftarrow \gamma) \leftarrow (\alpha \leftarrow \gamma)) \leftarrow ((\beta \leftarrow \alpha) \leftarrow \gamma)$
(E$_2$) $((\gamma \leftarrow \alpha) \leftarrow (\gamma \leftarrow \beta)) \leftarrow (\beta \leftarrow \alpha)$
(E$_3$) $(\alpha \quad \beta) \leftarrow \alpha$
(E$_4$) $(\alpha \quad \beta) \leftarrow \beta$
(E$_5$) $((\gamma \leftarrow (\alpha \quad \beta)) \leftarrow (\gamma \leftarrow \beta)) \leftarrow (\gamma \leftarrow \alpha)$
(E$_6$) $\alpha \leftarrow (\alpha \quad \beta)$
(E$_7$) $\beta \leftarrow (\alpha \quad \beta)$
(E$_8$) $(((\alpha \quad \beta) \leftarrow \gamma) \leftarrow (\beta \leftarrow \gamma)) \leftarrow (\alpha \leftarrow \gamma)$
(E$_9$) $(\gamma \leftarrow (\alpha \quad \beta)) \leftarrow ((\gamma \leftarrow \beta) \leftarrow \alpha)$
(E$_{10}$) $((\gamma \leftarrow \beta) \leftarrow \alpha) \leftarrow (\gamma \leftarrow (\alpha \quad \beta))$
(E$_{11}$) $((\beta \rightarrow \alpha) \quad \beta) \leftarrow \alpha$
(E$_{12}$) $(\quad \alpha \leftarrow \quad \beta) \leftarrow (\beta \leftarrow \alpha)$
(E$_{13}$) $((\beta \quad \gamma) \rightarrow \alpha) \leftarrow (\gamma \rightarrow (\beta \rightarrow \alpha))$
(E$_{14}$) $\neg(\beta \leftarrow \alpha) \leftarrow (\beta \rightarrow \alpha)$
(E$_{15}$) $(\alpha \leftarrow \beta) \leftarrow \quad (\beta \rightarrow \alpha)$
(E$_{16}$) $\quad \alpha \leftarrow ((\beta \rightarrow \beta) \leftarrow \alpha)$
(E$_{17}$) $((\beta \rightarrow \beta) \leftarrow \alpha) \leftarrow \quad \alpha$
(E$_{18}$) $\neg\alpha \leftarrow (\alpha \rightarrow (\beta \leftarrow \beta))$
(E$_{19}$) $(\alpha \rightarrow (\beta \leftarrow \beta)) \leftarrow \neg\alpha$

Let C_{HB} and $\overline{C}_{HB}$ be consequence operations for HB and $\overline{HB}$, respectively. Then, both logics are dual in the following sense: for any $\alpha,\beta \quad L_{HB}$

$$\alpha \rightarrow \beta \quad C_{HB}(\ ) \qquad \text{iff} \qquad \alpha \leftarrow \beta \quad \overline{C}_{HB}(\ )$$

Every set $\overline{T} = \overline{C}_{HB}(\overline{T})$ is called a $\overline{C}_{HB}$-theory. If moreover $\overline{T} \neq L_{HB}$, then $\overline{T}$ is a nontrivial $\overline{C}_{HB}$-theory. Rauszer also defines a prime $\overline{C}_{HB}$-theory: a nontrivial $\overline{C}_{HB}$-theory $\overline{T}$ is prime, if for any $\alpha,\beta \quad L_{HB}$

$$\alpha \quad \beta \quad \overline{T} \qquad \text{implies} \qquad \alpha \quad \overline{T} \ \text{ or } \ \beta \quad \overline{T}$$

An opposite implication follows directly from (E$_3$) and (E$_4$). It is an important observation that every prime $\overline{C}_{HB}$-theory is a complementation to the HB-language of some prime C_{HB}-theory, and every prime

$\overline{C}_{HB}$-theory is a complementation to the HB-language of some prime $\overline{C}_{HB}$-theory.

A symmetrical to the intuitionistic case verification shows, that for every consequence operation with (E_1), (E_2) and Modus Tollens the following Deduction Theorem holds:

$$\alpha \leftarrow \beta \quad \overline{T} \qquad \text{iff} \qquad \alpha \quad \overline{C}_{HB}(\overline{T}+\beta)$$

for any α,β L_{HB}. A $\overline{C}_{HB}$-theory $\overline{T}$ is maximal relatively to α, if

- $\alpha \quad \overline{T}$
- $\alpha \quad \overline{C}_{HB}(\overline{T}+\beta)$ for any $\beta \quad \overline{T}$.

A $\overline{C}_{HB}$-theory maximal relatively to some formula is a relatively maximal $\overline{C}_{HB}$-theory.

It is not surprising that every relatively maximal $\overline{C}_{HB}$-theory is a prime $\overline{C}_{HB}$-theory.

Also evidently, for any $\overline{C}_{HB}$-theory $\overline{T}$ and for any formula α not belonging to $\overline{T}$ there exists the $\overline{C}_{HB}$-theory maximal relatively to α (the Lindenbaum Lemma).

HB and $\overline{HB}$ are both deductive logics with such difference that the first logic derives true sentences from true sentences, while the second one derives false sentences from false sentences. The term "deductive logic" suggests that there exists some logic which is not deductive. Indeed, $\overline{HB}$ as a logic of falsehood can be the base for defining completely different kind of logic, a logic which is here called the reductive HB logic. The next section presents some facts concerning this new formal logic.

2. The elimination operation

In 1933 Tarski gave postulates which should be satisfied by an operation of the logical consequence. Since then it has been accepted that a deductive reasoning should fulfil the three well known Tarski conditions (see [9]).

It seems that similarly fundamental and obvious postulates can also be formulated in case of reductive reasoning.

Definition 2.1. Let X and Y be any sets of formulas of the language *L*. A function E:$2^L \rightarrow 2^L$ is an *elimination operation*, if

(A) E(X)   X

(B) X   Y   implies   E(X)   E(Y)

(C) E(X)   EE(X)

An operation E is *structural*, if for any endomorphism *e* of language *L* and for any X   L

(D) *e*(L−E(X))   L−E(L−*e*(L−X))

E is a *cofinitary* elimination operation, if for any X

(E) E(X) = ∩{E(Y): L−Y is a finite subset of L−X}

Since there exist two logical operations: consequence and elimination, we will distinguish two kinds of theories, C-theory and E-theory. And so, every set X=C(X) will be called a C-theory and Y=E(Y) an E-theory, respectively. The empty set is the smallest (trivial) E-theory and will be called an *insufficient* E-theory. Every nontrivial E-theory will be called *sufficient*. E(L) is the biggest E-theory.

Just as there exist many different consequence operations, also many different elimination operations can be defined. We intend to reconstruct for every consequence operation „its own" elimination operation, and by their means define one logic in its deductive-reductive form. However not every consequence operation „fits" every elimination operation.

An easy verification shows that, for C,E: $2^L \rightarrow 2^L$

- if C is a finitary, structural consequence operation and for any X, E(X)=L−C(L−X), then E is a cofinitary, structural elimination operation;
- if E is a cofinitary, structural elimination operation and for any X, C(X)=L−E(L−X), then C is a finitary, structural consequence operation.

Obviously, for any consequence and elimination operations, C and E, equalities: E(X)=L−C(L−X) and C(X)=L−E(L−X) are equivalent for any X.

Using the above observations, we are able to find the lacking elimination operation for HB logic. It is sufficient to put

$$E_{HB}(X) = L - \overline{C}_{HB}(L - X)$$

for any X L. The so connected two operations of consequence and elimination will be called *linked*.

$E_{HB}(X)$ is a set of these formulas which belong to X and cannot be inferred from the set $L-X$ by $\overline{C}_{HB}$. The meaning of this operation is as follows. Assume that we accept a set X. It means that other sentences, i.e. all sentences from $L-X$, can be treated as false. But obviously, there is also a logical inference between false sentences, and so if some sentences are false for us, then some other sentences have to be false for us, as well. All these sentences have to be rejected from X, and in result, they are not in $E_{HB}(X)$.

From the above definition it directly follows that every E_{HB}-theory is a complementation to the HB-language of some $\overline{C}_{HB}$-theory. It means that E_{HB}-theory is closed (maybe “open” would be a better term) under the following E-rule:

$$\alpha \quad E_{HB}(X), \qquad \text{whenever} \qquad \alpha \leftarrow \beta \quad E_{HB}(X) \quad \text{and} \quad \beta \quad E_{HB}(X).$$

The next E-rule for our logic has the form: α $E_{HB}(X)$ whenever $\neg\alpha$ $E_{HB}(X)$. All axioms for $\overline{HB}$, i.e. formulas (E_1)-(E_{19}), also constitute an axiom set for reductive HB. However, their application is quite different. Now,

$$\delta \quad E_{HB}(L)$$

for any axiom formula δ. Also in this point, there is full symmetry between deductive and reductive cases. A formula δ is an axiom for deductive logic if it belongs (even) to the smallest theory of this logic, the same δ belongs to every theory of a given logic. Similarly, a formula δ is an axiom for reductive logic if it does not belong (even) to the biggest theory of this logic, and so δ belongs to no theory of this logic.

The most interesting and important cases are the complementations of relatively maximal $\overline{C}_{HB}$-theories, which should be called relatively minimal E_{HB}-theories. Indeed, assume that a $\overline{C}_{HB}$-theory $\overline{T}$ is maximal relatively to α. Then, α $L-\overline{T}$ and α $L-\overline{C}_{HB}(L-(\overline{T}+\beta))$ for any β $L-\overline{T}$. In this way we obtain a definition of an E_{HB}-theory minimal relatively to α.

Definition 2.2. Let E be an elimination operation on the language *L*. An E-theory F is minimal relatively to α if

1. α F and
2. α E(F−β) for any β F.

An E-theory minimal relatively to some formula is relatively minimal.

Definition 2.3. An E_{HB}-theory F is prime if for any α,β

α β F iff α F and β F.

Of course, every relatively minimal E_{HB}-theory is prime. By E-axioms for disjunction (formulas (E_6)-(E_8)), we have

α β F iff α F or β F.

It means that if a formula set X is simultaneously a C_{HB}-theory and an E_{HB}-theory, then it is a prime C_{HB}-theory and a prime E_{HB}-theory.

Two propositions below confirm a deep duality between elimination and consequence operations.

Reduction Theorem. Let E be an elimination operation given by E-axioms (E_1) and (E_2), and Modus Tollens in the E-version. For any {α,β} X L_{HB}

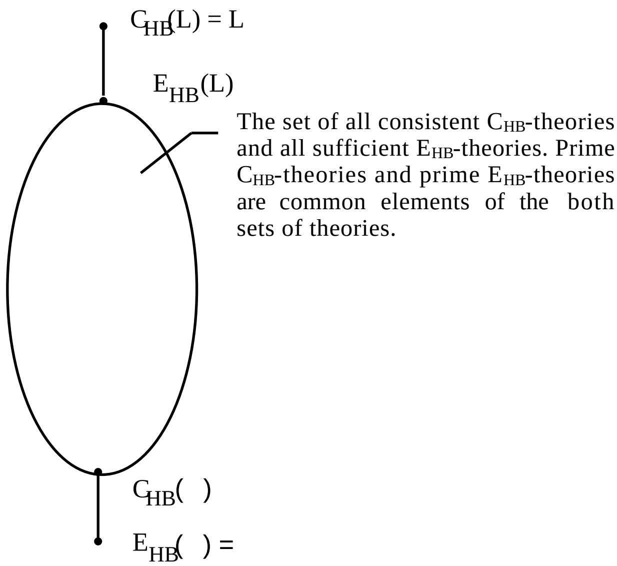
α ↼ β E(X) iff α E(X−β).

Proof. Assume that α↼β E(X), i.e. α↼β $\overline{C}(L-X)$. By Deduction Theorem for $\overline{C}$, it is equivalent to α $\overline{C}((L-X)+\beta)=\overline{C}(L-(X-\beta))$. Thus, α E(X−β). ■

Dual to Lindenbaum Lemma. For any E_{HB}-theory T and for any α T there exists an E_{HB}-theory T_0 minimal relatively to α such that T_0 T.

Proof. Directly from the Lindenbaum Lemma for $\overline{C}_{HB}$. ■

The diagram below presents the inclusion relation between all C_{HB}-theories and E_{HB}-theories:



Now we may introduce our main claim. If a logic is to formalize a valid inference, then it may not be limited only to the deductive part, since as it has been already shown, reductive reasonings are also valid. For this reason logic comprises two linked operations of consequence and elimination:

$$(C, E)$$

The logic thus understood admits both the reasonings which extend our system of beliefs with new accepted propositions, and the ones which eliminate no-longer accepted propositions.

In this paper the idea is realized on HB logic. Contrary to [7] the name of "Heyting-Brouwer logic" shall not refer solely to its deductive part but to full deductive-reductive form, i.e.

$$(C_{HB}, E_{HB})$$

As we have already presented, each of these operations has its own characteristic connectives. For consequence operation C_{HB} these are Heyting's connectives: implication, equivalence, negation, conjunction (right adjoint for implication). Here we may also include identity and necessity which do not appear in HB. Elimination operation E_{HB} is based on Brouwerian connectives: coimplication, coequivalence, weak negation, disjunction (left adjoint for coimplication). Moreover, Brouwerian connectives are non-identity and possibility (see [2]). Generally, it seems that Heyting's connectives are suitable to formalise deductive reasoning while Brouwerian connectives - reductive reasonings.

Two completely different (even contrary) epistemic roles of the operations yield the possibility to analyze of the set of accepted propositions by means of logic. Hence if, for example X is a set of beliefs, C(X) informs which sentences should be accepted whenever X is accepted. While E(X) determines how X should be limited because of $L-X$ is not accepted. That is why the set C(X) can be called *possible for X*, while E(X) can be called *necessary for X*. When X is inconsistent, for X everything is possible, while for insufficient X nothing is necessary.

Let us notice that we should be really satisfied only when E(X) = X = C(X), which is possible when our set of beliefs X is not only a C-theory but additionally a prime C-theory.

3. The presence of classical tautologies in HB

In [4] and [7] Rauszer presents many tautologies of Heyting-Brouwer logic. Here all HB-tautologies expressed without coimplication and weak negation will be called H-tautologies. We will determine B-tautologies in the following way: every HB-tautology not containing implication nor negation is a B-tautology (e.g. $\alpha \quad \alpha$). Additionally B-tautology is also every formula $\alpha \rightarrow \beta$, whenever $\alpha \leftarrow \beta \quad \overline{C}_{HB}(\quad)$ and neither α nor β contain implication and negation. We shall designate the set of all H-tautologies by H, while B will denote the set of B-tautologies. Evidently so determined sets, H and B, are not disjoint, see for instance formula $(\alpha \quad \beta) \rightarrow \beta$. Naturally there exist HB-tautologies belonging neither to H nor to B.

Some HB-tautologies have specific symmetry occurring between both types of connectives.

$$\vdash_{HB} (\neg\alpha \quad \beta) \to (\alpha \to \beta) \qquad \vdash_{HB} (\alpha \to \beta) \to (\quad \alpha \quad \beta)$$
$$\vdash_{HB} (\alpha \quad \neg\beta) \to (\alpha \leftarrow \beta) \qquad \vdash_{HB} (\alpha \leftarrow \beta) \to (\alpha \quad \beta)$$

Albeit Heyting's and Brouwerian cannot be mutually defined within HB, there exists mutual approximation. Similar symmetry is characteristic for both connectives of negation.

$$\vdash_{HB} \neg \quad \alpha \to \quad \alpha \qquad \vdash_{HB} \quad \alpha \to \alpha \qquad \vdash_{HB} \alpha \to \neg\neg\alpha \qquad \vdash_{HB} \neg\neg\alpha \to \quad \neg\alpha$$

It is not difficult to notice that none of the negations is privileged, both together yield full range of multiplied negations. Moreover, considering HB-tautologies wherein we have constant symmetry between both negations, none of them may be treated as "closer" to the classical negation. Both negations together constitute good and intuitive approximation of the classical negation in the intuitionistic reality.

Two of HB-tautologies there have special meaning. One of them is a typical H-tautology, a principle of inconsistency which forbids to accept both sentences whenever one is a negation of the other. Usually the Scotus law is expressed by means of conjunction appearing in the principle of inconsistency:

$$\vdash_{HB} (\alpha \quad \neg\alpha) \to \beta$$

Therefore, to accept any sentence together with its negation means that all other sentences are also accepted. We may say that this principle is essential for many deductive logics.

It turns out that there exists similarly fundamental principle for all reductive logics whose deductive counterparts obey the principle of inconsistency. In case of Heyting-Brouwer logic at the base this principle lies the law of excluded middle which bans on rejecting two such sentences of which one is a weak negation of the other. The analogue of the Scotus law for $\overline{\text{HB}}$ logic of falsehood is

$$\vdash - \quad \beta \leftarrow (\alpha \quad \alpha)$$

and thus

$$\vdash_{HB} \beta \to (\alpha \quad \alpha)$$

This means that simultaneous removal of a sentence together with its weak negation removes all other sentences as well.

Thus, the key formulas for HB in its complete, deductive-reductive form are:

$$\vdash_{HB} \neg(\alpha \quad \neg\alpha) \qquad\qquad \vdash_{HB} \alpha \qquad \alpha$$

It is straightforward that the principle of inconsistency reveals deductive character while the law of excluded middle is reductive. In more general terms, H-tautologies are formulas which determine the laws of deductive reasoning; B-tautologies correspond to countertautologies which define deductive logic of falsehood, thus represent the laws of reductive reasoning. Classical strengthening of HB results in intuitionistic implication and coimplication simply become classical implication and classical negation of classical implication, respectively. Obviously both negations become one and the same classical negation. Then, the set of all H- and B-tautologies gives entire set of classical tautologies and thus every classical tautology comes either from the set H or the set B. No such classical tautology could be found which would not appear in the intuitionistic logic of HB. Classical logic being typically ontological blurs the epistemic character of its tautologies. Nevertheless every classical tautology has deductive or reductive origin and thereby has its own epistemic value visible only in HB.

In the approach just presented, we look at each classical tautology in terms of its epistemic function in the deductive-reductive logic HB. The fact that some classical tautologies are not accepted in intuitionism took place because intuitionism (as any other logics) is reduced to its deductive part. Hence many intuitive tautologies are proclaimed non intuitionistic. Whereas every classical tautology can be found in intuitionism, although not necessarily in its deductive part. So, despite the intentions of its authors, Heyting's intuitionism having rejected so intuitive laws of classical logic as the law of excluded middle, violates the most basic intuitions. Concluding, logic in the deductive-reductive form not only augments the possibilities of our reasoning but also respects our basic intuitions.

Just as the principle of inconsistency is a tautology of intuitionism so is the law of excluded middle.

4. Heyting's intuitionistic logic

At the end, let us consider the traditional Heyting's propositional calculus (INT). Thus, the language is of the form:

$$L = (\mathrm{L}, \neg, \quad , \quad , \to)$$

An elimination operation for INT (E_{INT}) is defined by the following E-axioms (of course, all formulas (E_{INT-1})-(E_{INT-11}) are "forbidden" formulas):

(E_{INT-1}) $\neg(\neg(\alpha \to \beta) \to \alpha)$
(E_{INT-2}) $\neg(\neg(\neg(\gamma \to \alpha) \to \neg(\beta \to \alpha)) \to \neg(\neg(\gamma \to \beta) \to \alpha))$
(E_{INT-3}) $\neg((\alpha \quad \beta) \to \alpha)$
(E_{INT-4}) $\neg((\alpha \quad \beta) \to \beta)$
(E_{INT-5}) $\neg(\neg(\neg(\gamma \to (\alpha \quad \beta)) \to \neg(\gamma \to \beta)) \to \neg(\gamma \to \alpha))$
(E_{INT-6}) $\neg(\alpha \to (\alpha \quad \beta))$
(E_{INT-7}) $\neg(\beta \to (\alpha \quad \beta))$
(E_{INT-8}) $\neg(\neg(\neg(\alpha \quad \beta) \to \gamma) \to \neg(\beta \to \gamma)) \to \neg(\alpha \to \gamma))$
(E_{INT-9}) $\neg(\neg(\beta \to \neg\alpha) \to \neg(\alpha \to \neg\beta))$
(E_{INT-10}) $\neg(\beta \to (\alpha \to \alpha))$
(E_{INT-11}) $\alpha \quad \neg\alpha$

and the only E-rule

$$\alpha \quad E_{INT}(X), \quad \text{whenever} \quad \neg(\alpha \to \beta) \quad E_{INT}(X) \quad \text{and} \quad \beta \quad E_{INT}(X)$$

for α,β,γ L, X L.

As it was shown in [3], a semantics for E_{INT} is given by the class of the INT_E-models. Let an algebra $A = (\mathrm{A}, \neg, \cap, \quad , \to)$ be similar to the language L, a non empty set S be partially ordered by $\leq$, and for any s S, D_s be a subset of A. Then, a structure $M = (A, \{D_s : s \quad S\})$ is an INT_E-model, if for any a,b A and for any s S

a D_s	implies	for any	$t \geq s$	a D_t	
¬ a D_s	iff	for any	$t \leq s$	a D_t	
a ∩ b D_s	iff	a D_s	and	b D_s	
a b D_s	iff	a D_s	or	b D_s	
a → b D_s	iff	for any	$t \leq s$	(a D_t and b D_t)	

If for $\{\alpha\}$ X L,

α $E_M(X)$ iff for any h Hom(L, A), for any s S
h(α) D_s provided for any β X h(β) D_s

then, the following completeness theorem holds

α $E_{INT}(X)$ iff α $E_M(X)$ for any INT_E-model M

An easy verification shows that h(α $\neg\alpha$) D_s, for any INT_E-model M=(A,{D_s:s S}), for any s S and for any h Hom(L, A). It means that the analogue of the Scotus law for reductive part of INT is of the form

$$\neg(\beta \rightarrow (\alpha \quad \neg\alpha))$$

and a formula

$$\alpha \quad \neg\alpha$$

is satisfied in any point of any INT_E-model.

5. Conclusion

From two discussed in the paper reasons (one for the case of the HB logic, the second for INT) follows that

the law of excluded middle is an intuitionistic law

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