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THE NATURE OF INTUITIONISTIC POSSIBILITY

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Abstract

On the base of the classical logic the connectives of necessity and possibility have the equivalent positions in this sense that each of them is definable by the other one. The consequence of this fact is the possibility to define of the both modalities using the connective of identity. Thus, the connective of propositional identity defining the congruence of the propositional language has become the base of the reconstruction of necessity operator in some modal systems. Already in 1957 Greniewski [9] obtained system S5 on the base of the propositional calculus with identity. Next, Cresswell [3], [4] reconstructed S4 and S5 in a similar way. Systematic investigations of this problem were undertaken by Suszko within the broader frame of SCI (Sentential Calculus with Identity) programme [16], [17], [18].

The connections between modal connectives and the connective of identity settle some philosophical sense of both modalities. Something is necessary, if it is identical with a logical truth. If something is different from the negation of some logical truth, it is possible. It is however, the "ontological" characteristics typical for modalities defined on the base of the classical logic.

The settling of similar connections between intuitionistic identity and intuitionistic modalities uncover the philosophical sense of the "epistemic" necessity and possibility. Unfortunately, the symmetry between the classical and intuitionistic case is not entire. The acceptance of both equalities: $\Box = \neg\Diamond\neg$ and $\Diamond = \neg\Box\neg$, in the intuitionistic logic yields the classical versions of both modalities (i.e. the Law of Excluded Middle holds for all formulas for which necessity or possibility operator is main connective). Thus, there is a problem with defining the connective of intuitionistic possibility by the connective of identity.

In Sections 2 and 3 two "partial" solutions of the mentioned problem are presented. They are "partial" because the connective of possibility has undesirable properties. The second solution yields the strengthening of the necessity connective, too.

The main aim of this paper is obtained in Section 5, where a definition of the intuitionistic possibility is given. This definition uses the new connective of nonidentity, presented in Section 4. The new approach bases on the idea that some connectives like implication, negation, identity, necessity are "Heyting's" ones. The case of possibility is different. Together with coimplication, weak negation, and nonidentity, possibility belongs to the so-called "Brouwerian" connectives. The connectives of the first kind have such property that in Kripke's

style semantics their interpretations in each point of this semantics are related to the interpretations in next points. In the case of the “Brouwerian” connectives the interpretations in each point depend on the interpretations in previous points. In this sense, “Heyting's” connectives are interpreted from the point of view of the future, while the interpretations of the “Brouwerian” connectives are from the point of view of the past. The relations between nonidentity and possibility connectives allow to say that in temporal semantics something is possible in the moment *a*, if before *a* there was such moment *b* in which it was different from the contradiction. Thus, “something is possible” does not mean that it held once in the past. A broader philosophical remarks about the nature of intuitionistic possibility are contained in two Comments in the last Section.

1. Introduction

In this Section there is the main Suszko's results concerning the relations between identity and classical modalities will be recalled. The language for Sentential Calculus with Identity (SCI-language) is the language for classical sentential calculus with additional binary connective ≡, called “identity”

$$L_{SCI} = (L_{SCI}, \neg, \ , \ , \rightarrow, \leftrightarrow, \equiv)$$

The axiom set for SCI is the set of all axioms for classical sentential calculus enlarged by the set IDA (identity axioms) which consists of the following formulas

(A_1) $\alpha \equiv \alpha$

(A_2) $(\alpha \equiv \beta) \rightarrow (\neg\alpha \equiv \neg\beta)$

(A_3) $((\alpha \equiv \beta) \quad (\gamma \equiv \delta)) \rightarrow ((\alpha \ \S\ \gamma) \equiv (\beta \ \S\ \delta)), \quad \S \quad \{ \ , \ , \rightarrow, \leftrightarrow, \equiv\}$

(A_4) $(\alpha \equiv \beta) \rightarrow (\alpha \rightarrow \beta)$

Modus Ponens is the only inference rule. SCI-theory is Boolean, if it contains the set

$$WB = C_{SCI}(\{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \quad TFT\}),$$

where TFT is the set of truth-functional tautologies. Of course, WB is the smallest Boolean SCI-theory. Among Boolean SCI-theories two are interesting in a special way. First is

$$WT = C_{SCI}(\{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \quad C_{SCI}(\ )\}).$$

Let $L_m = (L_m, \Box, \neg, \;\; , \;\; , \to, \leftrightarrow)$. Let us consider two functions

f: $L_{SCI} \longrightarrow L_m$, such that

$f(\alpha) = \alpha$ if in α "$\equiv$" does not appear,

$f(\alpha \equiv \beta) = \Box(\alpha \leftrightarrow \beta)$, in other case;

g: $L_m \longrightarrow L_{SCI}$, such that

$g(\alpha) = \alpha$ if in α "$\Box$" does not appear,

$g(\Box\alpha) = (\alpha \equiv (\alpha \to \alpha))$, in other case.

Modal system S4 and SCI-theory WT are equivalent in the following sense

for any $\alpha \;\; L_{SCI} \;\; \alpha \;\; WT$ iff $f(\alpha) \;\; S4$
for any $\alpha \;\; L_m \;\; \alpha \;\; S4$ iff $g(\alpha) \;\; WT$

Similarly, if we consider an algebra

$$A = (A, \neg, I, \cap, \;\; , \to, \leftrightarrow, \circ)$$

where $(A, \neg, \cap, \;\; , \to, \leftrightarrow)$ is a Boolean algebra, I is a unary operator and $\circ$ is a binary operator, then the conditions, (I_1)-(I_4) given below, are equivalent to $(\circ_1)$-$(\circ_4)$, according to the translations

$$Ia = (a \circ 1) \quad \text{and} \quad (a \circ b) = I(a \leftrightarrow b) \qquad (1)$$

for any $a,b \;\; A$.

(I_1) $I1 = 1$ — $(\circ_1)$ $a \circ a = 1$

(I_2) $Ia \leq a$ — $(\circ_2)$ $(a \circ b) \leq (\neg a \circ \neg b)$

(I_3) $Ia = IIa$ — $(\circ_3)$ $((a \circ b) \cap (c \circ d)) \leq ((a \S c) \circ (b \S d))$, $\S \;\; \{\cap, \;\; , \to, \leftrightarrow, \circ\}$

(I_4) $I(a \cap b) = Ia \cap Ib$ — $(\circ_4)$ $(a \circ b) \leq (a \to b)$

Due to the form of IDA, it can be easily seen that Boolean algebra with binary operator satisfying $(\circ_1)$-$(\circ_4)$ is an algebraic model for WT. I satisfying the conditions (I_1)-(I_4) is an interior operator and hence A is a topological Boolean algebra. The same kind of algebra is given by satisfying $(\circ_1)$-$(\circ_4)$. $C = \neg I \neg$ is the closure operator, i.e.

(C_1) C0 = 0
(C_2) $a \leq Ca$
(C_3) $Ca = CCa$
(C_4) $C(a \quad b) = Ca \quad Cb$,

it is possible to define the translations between C and $\circ$ of the form: $Ca = \neg(a \circ 0)$ and $(a \circ b) = \neg C \neg (a \leftrightarrow b)$ for any a,b A. Thus, on the base of classical sentential calculus the both modal connectives of kind S4 are definable by propositional identity.

The second interesting SCI-theory is WH, i.e. the smallest Boolean SCI-theory where all substitutions of the formula are included (the so-called Greniewski's axiom [9])

(A_5) $((\alpha \equiv \beta) \equiv 0) \quad ((\alpha \equiv \beta) \equiv 1)$

using the same functions as in the case of S4, we get that

for any $\alpha \quad L_{SCI} \qquad \alpha \quad$ WH iff $f(\alpha) \quad$ S5,

for any $\alpha \quad L_m \qquad \alpha \quad$ S5 iff $g(\alpha) \quad$ WH.

In turn, additional conditions for semi-simple topological algebras now have the form

(I_5) $B = \{Ia: a \quad A\} = \{0, 1\}$ and ($\circ_5$) $a \circ b = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$

Obviously (I_5) can be replaced by (C_5) $T = \{Ca: a \quad A\} = \{0, 1\}$. Just defined ($I_5$) and ($\circ_5$) are equivalent respectively to the well known translation equalities.

2. Intuitionistic Sentential Calculus with Identity

The lack of the mutual definability of both intuitionistic modalities yields that there exist different strategies in extending intuitionistic logic by necessity and possibility connectives. One of them is to introduce “box” as a primitive operator and to define “diamond” by the equality: $\Diamond = \neg\Box\neg$. But what seems more popular is another approach introducing both modalities independently by appropriate axioms.

In this section we consider the first solution, elaborated mainly by Font [6], [7], [8].

In [7] Font defines five modal extensions of intuitionistic logic: IM4, IM4W, IM4M, IM4S and IM5;

IM4: $\Box\alpha \to \alpha$
$\Box\alpha \to \Box\Box\alpha$
$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
$\alpha \vdash \Box\alpha$ (The rule of Necessitation)

4W: $\neg\Box\neg\Box\neg\alpha \to \neg\neg\Box\neg\alpha$
4M: $\neg\Box\alpha \to \Box\neg\Box\alpha$
4S: $(\Box\alpha \to \Box\beta) \to \Box(\Box\alpha \to \beta)$
5: $\Box\alpha \quad \Box\neg\Box\alpha$

IMX = IM4+X for X = 4W,4M,4S,5. In every Font's system the possibility operator is given by $\Diamond = \neg\Box\neg$.

The algebraic model for IM4 is so-called tpBa (topological pseudo-Boolean algebra), i.e. the Heyting algebra with two additional unary operators: I satisfying the conditions (I_1)-(I_4) and $\delta = \neg I \neg$. Obviously, δ is not a closure operator, because $\delta(a \quad b) \leq \delta a \quad \delta b$ does not hold ([8]).

A tpBa is a model for IMX (X = 4W, 4M, 4S, 5), if moreover

(I_{4W}): $\neg I \neg I \neg a = \neg\neg I \neg a$
(I_{4M}): $\neg I a = \neg I a$
(I_{4S}): $(Ia \to Ib) = I(Ia \to b)$
$(I_{5'})$: $Ia \quad I \neg I a = 1$

hold, respectively. In [8] it is proved the equivalence of (I_5) and $(I_{5'})$.

Applying the translation (1) given in the previous section we may also define tpBa as a Heyting algebra with a new binary operation $\circ$ satisfying $(\circ_1)$-$(\circ_4)$. Strengthening thus obtained algebra respectively by

$(\circ_{4W})$: $\neg((\neg a \circ 1) \circ 0) = \neg\neg(\neg a \circ 1)$
$(\circ_{4M})$: $\neg(a \circ 1) = ((a \circ 1) \circ 0)$

$(\circ_{4S})$: $((a \circ 1) \to (b \circ 1)) = (((a \circ 1) \to (b \circ 1)) \circ 1)$

and $(\circ_5)$ we get models for the remaining systems. Thus, the "circle" operation enables us to express the weak monadicity, monadicity, strong monadicity and simplicity of topological Heyting algebras.

Just defined Heyting algebras with "circle" operation satisfying the above mentioned conditions are models for the ISCI-theories presented below. Notice that Intuitionistic Sentential Calculus with Identity (ISCI) is built on the same SCI-language L_{SCI}, which IDA extended the axiom set for intuitionistic sentential calculus. More about ISCI can be found in [10], [11], [12].

Using two functions "f'" and "g" given in the previous Section we obtain the equivalence of IM4 and WPT = $C_{ISCI}(\{\alpha\equiv\beta : \alpha \leftrightarrow \beta)$ $C_{ISCI}(\quad)\})$. In the same sense the next intuitionistic modal systems discussed by Font are equivalent to WPT extended by

(A_{4W}) $\neg((\neg\alpha \equiv 1) \equiv 0) \equiv \neg\neg(\neg\alpha \equiv 1)$
(A_{4M}) $\neg(\alpha \equiv 1) \to ((\alpha \equiv 1) \equiv 0)$
(A_{4S}) $((\alpha \equiv 1) \to (\beta \equiv 1)) \to (((\alpha \equiv 1) \to (\beta \equiv 1)) \equiv 1)$

and (A_5), respectively.

In the end let us notice that although the two-elementness of the set of all open elements of tpBa implies the two-elementness of the set of all closed elements of the same algebra, the converse of this implication does not hold. Thus, in the case of topological Heyting algebras the equivalence of (I_5) and (C_5) does not hold, either.

3. Heyting-Brouwer Sentential Calculus with Identity

Now, let us consider another approach to intuitionistic modal logics which has been developed by Bull (e.g. [2]), Ono (e.g. [13]), Fisher-Servi (e.g. [1], [5]), Wolter (e.g. [19]) and others.

To the axiom set for the intuitionistic sentential calculus we add

$(\Box_1)$ $\Box\alpha \to \alpha$ $(\Diamond_1)$ $\alpha \to \Diamond\alpha$
$(\Box_2)$ $\Box\alpha \to \Box\Box\alpha$ $(\Diamond_2)$ $\Diamond\Diamond\alpha \to \Diamond\alpha$
$(\Box_3)$ $(\Box\alpha \quad \Box\beta) \to \Box(\alpha \quad \beta)$ $(\Diamond_3)$ $\Diamond(\alpha \quad \beta) \to (\Diamond\alpha \quad \Diamond\beta)$
$(\Box\Diamond_1)$ $\Diamond(\alpha \to \beta) \to (\Box\alpha \to \Diamond\beta)$
$(\Box\Diamond_2)$ $(\Diamond\alpha \to \Box\beta) \to \Box(\alpha \to \beta)$

$R_{\Box}$ $(\alpha \to \beta) \vdash (\Box\alpha \to \Box\beta)$ $R_{\Diamond}$ $(\alpha \to \beta) \vdash (\Diamond\alpha \to \Diamond\beta)$

The obtained system will be called cIS4 (connected intuitionistic S4). cIS4 without ($\Box\Diamond_1$) and ($\Box\Diamond_2$) will be called IS4 (cf. [1]).

We will try to find the system equivalent to cIS4 among the theories of Heyting-Brouwer Sentential Calculus with Identity (HBSCI). Our new language is of the form

$$L_{\mathrm{HBSCI}} = (L_{\mathrm{HBSCI}}, \neg, \ \ , \ \ , \ \ , \to, \leftharpoondown, \leftrightarrow, \equiv)$$

Two new connectives: the weak negation and the coimplication $\leftharpoondown$ are characterized though the following axioms

$\alpha \to (\beta \quad (\alpha \leftharpoondown \beta))$
$(\alpha \leftharpoondown \beta) \to \quad (\alpha \to \beta)$
$((\alpha \leftharpoondown \beta) \leftharpoondown \gamma) \to (\alpha \leftharpoondown (\beta \quad \gamma))$
$\neg(\alpha \leftharpoondown \beta) \to (\alpha \to \beta)$
$(\alpha \to (\beta \leftharpoondown \beta)) \to \neg\alpha$
$\neg\alpha \to (\alpha \to (\beta \leftharpoondown \beta))$
$((\beta \to \beta) \leftharpoondown \alpha) \to \quad \alpha$
$\quad \alpha \to ((\beta \to \beta) \leftharpoondown \alpha)$

The next inference rule after Modus Ponens is: $\alpha \vdash \neg \quad \alpha$. For more about Heyting-Brouwer logic see Rauszer [14], [15].

Since neither $(\alpha\equiv\beta) \to (\quad \alpha\equiv \quad \beta)$ nor $((\alpha\equiv\beta) \quad (\gamma\equiv\delta)) \to ((\alpha\leftharpoondown\gamma)\equiv(\beta\leftharpoondown\delta))$ is a tautology of HBSCI, invariant identity axioms for weak negation and coimplication warranting the congruence of identity are of the form

$((\alpha\equiv\beta) \equiv (\alpha\to\alpha)) \to ((\quad \alpha\equiv \quad \beta) \equiv (\alpha\to\alpha))$
$(((\alpha\equiv\beta) \equiv (\alpha\to\alpha)) \quad ((\gamma\equiv\delta) \equiv (\alpha\to\alpha))) \to (((\alpha\leftharpoondown\gamma) \equiv (\beta\leftharpoondown\delta)) \equiv (\alpha\to\alpha))$.

The algebraic model for

$$\mathrm{WHBT} = C_{\mathrm{HBSCI}}(\{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \quad C_{\mathrm{HBSCI}}(\quad)\})$$

is a Heyting-Brouwer algebra with additional binary operation satisfying the already known conditions ($\circ_1$)-($\circ_4$) and, moreover,

($\circ_{21}$) $((a \circ b) \circ 1) \leq ((\sim a \circ \sim b) \circ 1)$
($\circ_{31}$) $(((a \circ b) \circ 1) \cap ((c \circ d) \circ 1)) \leq (((a \leftharpoonup c) \circ (b \leftharpoonup d)) \circ 1)$.

The defined algebra will be called strongly topological Heyting-Brouwer (stHBa). After [15] we say that (A, ¬, ~, ∩, , →, $\leftharpoonup$) is a Heyting-Brouwer algebra (HB-algebra), if

(A, ¬, ∩, , →) is a Heyting algebra and
(A, ~, ∩, , $\leftharpoonup$) is a Brouwer algebra.

Thus the following connections hold

(a) $x \leq a \to b$ iff $a \cap x \leq b$
(b) $a \leftharpoonup b \leq x$ iff $a \leq b$ x
(c) $1 = a \to a$ the greatest element of A
(d) $0 = a \leftharpoonup a$ the least element of A
(e) $\neg a = a \to 0$ the ∩-complement of a
(f) $\sim a = 1 \leftharpoonup a$ the -complement of a
(g) $a \to b = 1$ iff $a \leq b$ iff $a \leftharpoonup b = 0$.

Proposition 3.1. In every stHBa A, for any a A
(a) $(a \circ 1) = (\sim a \circ 0)$;
(b) $(a \circ 0) = (\neg a \circ 1)$;
(c) $(a \circ 0) \leq (\sim a \circ 1)$;
(d) $(a \circ 1) = (\neg \sim a \circ 1)$;
(e) $(a \circ 1) = \neg \sim (a \circ 1)$.

Proof. (a). Successively by ($\circ_3$), ($\circ_{21}$), ($\circ_4$) and ($\circ_2$) $(a\circ 1) \leq ((a\circ 1) \cap (1\circ 1)) \leq ((a\circ 1)\circ(1\circ 1)) = ((a\circ 1)\circ 1) \leq ((\sim a\circ\sim 1)\circ 1) \leq (\sim a\circ 0) \leq (\neg\sim a\circ 1) \leq (a\circ 1)$. (b). By ($\circ_2$) and (a) $(a\circ 0) \leq (\neg a\circ 1) = (\sim\neg a\circ 0) \leq (a\circ 0)$. (c). Proof analogous to the first steps of the proof for (a) it is sufficient to replace "1" with "0". (d) follows directly from (a) and ($\circ_2$). (e). By ($\circ_3$), (d) and ($\circ_4$) $\neg\sim(a\circ 1) \leq (a\circ 1) = ((a\circ 1)\circ 1) = (\neg\sim(a\circ 1)\circ 1)$ $(\neg\sim(a\circ 1) \leq (a\circ 1)$. □

An easy verification shows that in each algebraic model for WHBT there are definable the interior and closure operators. Indeed, $Ia = (a\circ 1)$ and $Ca = \sim(a\circ 0)$ (for a A) satisfy, respectively, the conditions (I_1)-(I_4) and (C_1)-(C_4). Notice that due to (I_4) and (C_4) the both operators are monotonic. Moreover, from 3.1 it follows

Proposition 3.2. In every stHBa A with $(a \circ 1) = \mathrm{I}a$ and $\neg(a \circ 0) = \mathrm{C}a$, for any a A

(a) $\mathrm{I}a = \mathrm{I}\neg{\sim}a$;
(b) $\mathrm{C}a = \mathrm{C}{\sim}\neg a$;
(c) $\mathrm{I}a = \neg{\sim}\mathrm{I}a$;
(d) $\mathrm{C}a = {\sim}\neg\mathrm{C}a$.

Proof. (a) follows from 3.1(d). (b). By (2) and 3.1.(a) $\mathrm{C}{\sim}\neg a = {\sim}(\neg{\sim}a \circ 0) \leq {\sim}(a \circ 0) = \mathrm{C}a$. The opposite inequality follows from the monotonicity of C. (c) directly follows from 3.1(e). (d). Successively by 3.1(b), 3.1(e), 3.1.(b) $\neg\mathrm{C}a = {\sim}\neg{\sim}(a \circ 0) = {\sim}\neg{\sim}({\sim}a \circ 1) = {\sim}(\neg a \circ 1) = {\sim}(a \circ 0) = \mathrm{C}a$. ☐

Proposition 3.3. In every stHBa A with $(a \circ 1) = \mathrm{I}a$ and $\neg(a \circ 0) = \mathrm{C}a$, for any a A

(a) $\mathrm{I}a = \neg\mathrm{C}{\sim}a$;
(b) $\mathrm{C}a = {\sim}\mathrm{I}\neg a$;
(c) $\mathrm{I}\neg a = \neg\mathrm{C}a$;
(d) $\mathrm{C}{\sim}a = {\sim}\mathrm{I}a$.

Proof. (a). By 3.1(e) and 3.1(a) $\mathrm{I}a = (a \circ 1) = \neg{\sim}(a \circ 1) = \neg{\sim}({\sim}a \circ 0) = \neg\mathrm{C}{\sim}a$. (b). By 3.1(b) $\mathrm{C}a = {\sim}(a \circ 0) = {\sim}(\neg a \circ 1) = {\sim}\mathrm{I}\neg a$. (c). By 3.1(b) and 3.1(e) $\neg\mathrm{C}a = \neg{\sim}(a \circ 0) = \neg{\sim}(\neg a \circ 1) = (\neg a \circ 1) = \mathrm{I}\neg a$. (d). By 3.1(b) and 3.1(e) $\mathrm{C}{\sim}a = {\sim}({\sim}a \circ 0) = {\sim}(\neg{\sim}a \circ 1) = {\sim}(a \circ 1) = {\sim}\mathrm{I}a$. ☐

From now on the notion “stHBa” will be simultaneously the name for:

1. Heyting-Brouwer algebra with additional binary operation $\circ$ satisfying the conditions $(\circ_1)$-$(\circ_4)$ and $(\circ_{21})$-$(\circ_{31})$,

2. Heyting-Brouwer algebra with two additional unary operators I and C satisfying the conditions (I_1)-(I_4), (C_1)-(C_4) and 3.3(a)-(b).[1]

The next theorem settles strong properties of both topological operators.

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[1] Assuming $(a \circ b) = \mathrm{I}(a \leftrightarrow b)$ for any a,b A we can infer $(\circ_1)$-$(\circ_4)$ only from (I_1)-(I_4). For proving $(\circ_{21})$-$(\circ_{31})$ it is necessary to apply 3.2(a) and 3.2(c) which follow from 3.3(a)-(b).

Proposition 3.4. In every stHBa A, for any a A

(a) $\neg \mathrm{I}a = {\sim}\mathrm{I}a$;
(b) $\neg \mathrm{C}a = {\sim}\mathrm{C}a$.

Proof. (a) $\neg \mathrm{I}a \leq {\sim}\mathrm{I}a \leq \neg\neg {\sim}\mathrm{I}a \leq \neg \mathrm{I}a$ (by 3.2(c)). (b) $\neg \mathrm{C}a \leq {\sim}\mathrm{C}a = {\sim\sim}\neg \mathrm{C}a \leq \neg \mathrm{C}a$ (by 3.2(d)). □

The connections between two topological operators, are expressed by the following equalities:

$$\mathrm{I}a = \$\mathrm{C}{\sim}a \qquad \text{and} \qquad \mathrm{C}a = \$\mathrm{I}\neg a, \tag{2}$$

with \$ $\{\neg, {\sim}\}$ and a A.

The next theorem establishes two axioms of IS4:
($\Box\Diamond_1$) and ($\Box\Diamond_2$).

Proposition 3.5. In every stHBa A, for any a,b A

(a) $\mathrm{C}(a \to b) \leq \mathrm{I}a \to \mathrm{C}b$,
(b) $\mathrm{C}a \to \mathrm{I}b \leq \mathrm{I}(a \to b)$.

Proof. Because of (2), the proof of (a) is a repetition of the proof for the case with $\mathrm{C} = \neg \mathrm{I} \neg$, see [8]. (b). In every HB-algebra A, $a \to b \leq {\sim}a$ b for any a A. Thus, $\mathrm{C}a \to \mathrm{Ib} \leq {\sim}\mathrm{C}a$ $\mathrm{I}b = \neg \mathrm{C}a$ $\mathrm{I}b = \mathrm{I}\neg a$ $\mathrm{I}b \leq \mathrm{I}(\neg a$ $b)$ $\leq \mathrm{I}(a \to b)$.

In [8] Font considers three extensions of tpBa (topological pseudo-Boolean algebra). They are: weakly monadic, monadic, strongly monadic and semi-simple tpBa. Still in [8], there are given several conditions (some of them are equivalent) for each kind of tpBa. We choose the two following:

1. $\mathrm{C}\neg \mathrm{C}a = \neg \mathrm{C}a$, for weak monadicity;
2. $\mathrm{CI}a = \mathrm{I}a$, for simplicity.

Because of (2) every stHBa is tpBa. Example 1 from Appendix proves that stHBa does not satisfy 1. Thus, stHBa is neither weakly monadic nor stronger than tpBa. Simultaneously, in the case of stHBa semi-simple all kinds of monadicities collapse to the one, only. Indeed, every semisimple tpBa is weakly monadic, monadic and strongly monadic. Moreover

Proposition 3.6. In every stHBa A the following conditions are equivalent

(a) C¬Ca = ¬Ca for any a A;
(b) CIa = Ia for any a A.

Proof. For the proof that (b) implies (a), see [8]. Now assume (a). CI*a* = C¬C~*a* = ¬C~*a* = I*a*. ☐

Thus, we can conclude that stHBa is a model for the system properly stronger than cIS4. Moreover, the axiomatic extensions of this new system by any of the following formulas: 4W, 4M, 4S, 5 result in the same system properly stronger than cIS5 = cIS4+5.

4. The nonidentity connective

From previous sections it follows that defining intuitionistic possibility with identity connective brings about undesirable strengthening of both modalities and too strong connection between them. The key idea of this Section is the complete independence of the two intuitionistic modalities. Thus, the identity connective will be used for defining of the necessity connective only. The same, the definition of the intuitionistic possibility will employ a new connective of "nonidentity" ≢ analogous to identity. Our approach is designed on the construction of the coimplication as the connective dual to implication. (cf. (a) and (b) from section 3). Neither coimplication is any (weak or intuitionistic) negation of the implication, nor nonidentity is any negation of identity.

The Heyting-Brouwer Sentential Calculus with Identity and Nonidentity (HBSCIN) is built on the HBSCIN-language, i.e. on the algebra

$$L_{\mathrm{HBSCIN}} = (L_{\mathrm{HBSCIN}}, \neg, \ , \ , \ , \rightarrow, \leftharpoondown, \leftrightarrow, \rightleftharpoons, \equiv, \not\equiv),$$

with $\alpha \rightleftharpoons \beta = (\alpha \leftharpoondown \beta) \quad (\beta \leftharpoondown \alpha)$.

The nonidenity connective axiom set NDA is dual to IDA and contains all substitutions of the following formulas

(B_1) $\neg(\alpha \not\equiv \alpha)$

(B$_2$) (α ≋ β) → (α ≋ β)
(B$_3$) ((α § γ) ≋ (β § δ) → ((α ≋ β) (γ ≋ δ)), § { , ,↼,⇌, ≋ }
(B$_4$) (α ↼ β) → (α ≋ β)

The rules are the same as in the case of HBSCI. The definition of the consequence operation C_{HBSCIN} is standard.

For the identity connective we accept the axiom set IDA, only. The result of such decision is that the relation $_T$ defined as follows

$$\alpha \quad _T \beta \qquad \text{if and only if} \qquad (\alpha \equiv \beta) \quad T, \tag{3}$$

for any α,β L_{HBSCIN} and for any HBSCIN-theory T, is not the congruence of the whole language L_{HBSCIN}. Now $_T$ is the congruence of the matrix $L_{HBSCIN*}$,T , where $L_{HBSCIN*}$ is the HBSCIN-language in which the “Brouwerian” connectives (i.e. ,↼,⇌,≋) are not treated as algebraic operations. It means that the conditions

$$[\ \alpha]\ _T = \quad _T [\alpha]\ _T \tag{4}$$

$$[\alpha \S \beta]\ _T = [\alpha]\ _T \S\ _T [\beta]\ _T \tag{5}$$

for § {↼, ⇌ , ≋}; are not required. Of course $[\alpha]_T$ is the equivalence class of α by the relation $_T$. Brouwerian connectives can appear in the formulas, only. Thus the relation $_T$ is “blind” on the mentioned connectives.

Analogously, the relation ≈ defined by condition

$$\alpha \approx_T \beta \qquad \text{if and only if} \qquad (\alpha \text{ ≋ } \beta) \quad \overline{T},\ ^2 \tag{6}$$

for any α,β L_{HBSCIN} and for any HBSCIN-theory T, as the previous relation, is not the congruence of the whole language L_{HBSCIN}. Our new relation is the congruence of the matrix $L_{HBSCIN**}$, T , where $L_{HBSCIN**}$ is the HBSCIN-language in which the “Heyting” connectives (¬, →, ↔, ≡) are not treated as an algebra's operations. Now, the following equalities

---

[2] $\overline{T}$ is a complement T to the whole language. Thus our case $\overline{T} = L_{HBSCIN} -$ T, cf. [15].

$$[\neg\alpha]_{\approx_T} = \neg_{\approx_T}[\alpha]_{\approx_T} \qquad (7)$$

$$[\alpha \S \beta]_{\approx_T} = [\alpha]_{\approx_T} \S_{\approx_T} [\beta]_{\approx_T} \qquad (8)$$

for § {→, ↔, ≡} are not guaranteed. In this case the relation $\approx_T$ is "blind" on the "Heyting" connectives. For the proof of symmetry of $\approx_T$, see Example 2 in Appendix.

It is easy to see that adding to IDA+NDA the axiom

$$(\alpha \equiv \beta) \leftrightarrow \neg(\alpha \not\approx \beta) \qquad (9)$$

the two relations coincide to the same congruence of the HBSCIN-language.

Now, we present the semantics for HBSCIN. Let A = (A, ¬, ~, ∩, , →, ↼, ∘, ·) be an algebra similar to HBSCIN-language. For each D A let be the relation of the first kind, i.e. a congruence of a matrix (A, ¬, ∩, , →, ∘), D and ≈ be the relation of second kind, i.e. a congruence of (A, ~, ∩, , ↼, ·), D .

Let us consider two classes of matrices.

$\mathbf{M_1}$ = { A/ , D/ : D A and is the relation of first kind}and
$\mathbf{M_2}$ = { A/≈, D/≈ : D A and ≈ is the relation of second kind}.

Let X and Y be the sets of indexes of elements of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Thus

M_1 = { A_x, D_x : x X}
M_2 = { A_y, D_y : y Y}

On both sets of indexes the relations $\leq_X$ and $\leq_Y$ are defined as follows:

- for any x_1,x_2 X $x_1 \leq_X x_2$ iff D_1 D_2, where $D_{xi}=D_i/_i$ for i {1,2},
- for any y_1,y_2 Y $y_1 \leq_Y y_2$ iff D_1 D_2, where $D_{yi}=D_i/\approx_i$ for i {1,2}.

Notice that for any α L_{HBSCIN} and for any x X there exists y Y such that

$$[\alpha]_x \quad D_x \quad \text{iff} \quad [\alpha]_y \quad D_y \qquad (10)$$

Indeed, (10) is a consequence of the fact that each of the relations is a congruence of the corresponding matrix.

Definition 4.1. A subset *M* of $\mathbf{M}_1$ $\mathbf{M}_2$ will be an HBSCIN-model, if for any *x* X, *y* Y and *a*,*b* A

(0) if $[a]_z$ D_z, then $t \geq_Z z$ $[a]_t$ D_t;
(1) $\neg [a]_x$ D_x iff $t \geq_X x$ $[a]_t$ D_t;
(2) $\sim [a]_y$ D_y iff $t \leq_Y y$ $[a]_t$ D_t;
(3) $[a]_z \cap [b]_z$ D_z iff $[a]_z$ D_z and $[b]_z$ D_z;
(4) $[a]_z$ $[b]_z$ D_z iff $[a]_z$ D_z or $[b]_z$ D_z;
(5) $[a]_x \rightarrow [b]_x$ D_x iff $t \geq_X x$ ($[a]_t$ D_t or $[b]_t$ D_t);
(6) $[a]_y \leftharpoondown [b]_y$ D_y iff $t \leq_Y y$ ($[a]_t$ D_t and $[b]_t$ D_t);
(7) $[a]_x \circ [b]_x$ D_x iff $t \geq_X x$ $[a]_t = [b]_t$;
(8) $[a]_y \cdot [b]_y$ D_y iff $t \leq_Y y$ $[a]_t \neq [b]_t$,

with (*z*, Z) = (*x*, X), (*y*, Y).

Definition 4.2. For any α L_{HBSCIN}, B L_{HBSCIN}
α C_M(B) iff for any h Hom (L_{HBSCIN}, A) for any z Z
k_z o h (α) D_z provided that k_z o h (β) D_z for all β B,

with (z, Z) = (x, X), (y, Y) and k_z - the canonical homomorphism *A* onto A_z.

Proposition 4.1. Let α L_{HBSCIN}, B L_{HBSCIN}. The following conditions are equivalent

(a) α C_{HBSCIN} (B),
(b) α C_M(B) for any HBSCIN-model *M*.

Proof. **Soundness**. As an example one axiom B_4 will be checked, only. Assume that for some HBSCIN-model *M*, for some h Hom(L_{HBSCIN}, *A*) (*A* is the base of the construction of *M*) for some *x* X and k_y,
k_y o h((α ↽ β) → (α ≹ β)) D_y. Because of (10)
k_x((h(α) ↽ h(β)) → (h(α) · h(β))) D_x for some x X. By 4.1(5)
$x_1 \geq_X x$ k_{x1}(h(α) ↽ h(β) D_{x1} and k_{x1} (h(α) · h(β)) D_{x1}. By (10)
y Y k_y(h(α) ↽ h(β)) D_y and k_y(h(α) · h(β)) D_y. By 4.1(6),(8)

y Y ($y_1 \leq_Y y$ ($k_{y1}(h(\alpha))$ D_{y1} and $k_{y1}(h(\beta))$ D_{y1}) and
$t \leq_Y y$ $k_t(h(\alpha)) = k_t(h(\beta))$)).

There is a contradiction in k_{y1}. Example 3 (see Appendix) shows the falsification of the formula $\neg(\alpha \not\approx \beta) \rightarrow (\alpha \equiv \beta)$.

Completeness. Assume that for some α L_{HBSCIN}, and B L_{HBSCIN} α $L_{HBSCIN}(B)$. Let T_0 be a relatively maximal HBSCIN-theory containing the set B but α T_0. The HBSCIN-model satisfying all formulas from B and falsifying α is composed of two cones of matrices. Each of them is built on the base of HBSCIN-language and the set of all relatively maximal HBSCIN-theories containing T_0, in the first case and in the second case - contained in T_0. All matrices from the first cone are divided by relation (3), for the elements of the second cone we used the relation (6). The verification of (0), (1), (3), (4), (5) from definition 4.1 is standard. For (2) and (6) see [15], for (7) see [10]. We check the last condition.

Assume that $[\alpha]_{\approx T} \not\approx [\beta]_{\approx T}$ $T_{\approx T}$. Thus $[\alpha \not\approx \beta]_{\approx T}$ $T_{\approx T}$, so $\alpha \not\approx \beta$ $\overline{T}$ and for any $\overline{T}'$ $\overline{T}$ $\alpha \not\approx \beta$ $\overline{T}'$. It means that for any $\overline{T}'$, $\overline{T}$ $\overline{T}'$ $\alpha \approx_{T'} \beta$, so for any T' T $[\alpha]_{\approx_{T'}} = [\beta]_{\approx_{T'}}$. On the other hand, it is sufficient to proceed in the opposite direction. □

From some purposes the semantics given through epistemic valuations seems more useful. Below, we present such a tool for HBSCIN.

Definition 4.3. Let X be any non empty set. With every x X let us relate two functions f_x^1, f_x^2 and two partial orders $\leq_x^1$, $\leq_x^2$. For any x X

(a) $f_x^1: L_{HBSCIN} \times X_x^1 \dashrightarrow \{0,1\}$, with $X_x^1 = \{y \;\; X\}: x \leq_x^1 y\}$;
(b) $f_x^2: L_{HBSCIN} \times X_x^2 \dashrightarrow \{0,1\}$, with $X_x^2 = \{y \;\; X\}: y \leq_x^2 x\}$.

$\mathbf{f}_X = \{f_x^i: x \;\; X \text{ and } i=\{1,2\}\}$ will be called an epistemic valuation for HBSCIN, if for any α,β L_{HBSCIN} and for any x,y X and i={1,2}

(+)	$f_x^1(\alpha,y) = f_y^2(\alpha,y)$	and	$f_x^2(\alpha,y) = f_y^1(\alpha,y)$;	
(0)	y,z X_x^i: if $y \leq_x^i z$,	then	$f_x^i(\alpha,y) \leq f_x^i(\alpha,z)$;	
(1)	y X_x^1: $f_x^1(\neg\alpha,y) = 1$	iff	$t \geq_x^1 y$ $f_x^1(\alpha,t)=0$;	
(2)	y X_x^2: $f_x^2(\;\alpha,y) = 1$	iff	$t \leq_x^2 y$ $f_x^2(\alpha,t)=0$;	
(3)	y X_x^i: $f_x^i(\alpha \;\; \beta,y) = 1$	iff	$f_x^i(\alpha,y)=1$ and $f_x^i(\beta,y)=1$;	
(4)	y X_x^i: $f_x^i(\alpha \;\; \beta,y) = 1$	iff	$f_x^i(\alpha,y)=1$ or $f_x^i(\beta,y)=1$;	

(5) $y\ \ X_x^1$: $f_x^1(\alpha \rightarrow \beta, y) = 1$ iff $t \geq_x^1 y$ $(f_x^1(\alpha,t)=0$ or $f_x^1(\beta,t)=1)$;
(6) $y\ \ X_x^2$: $f_x^2(\alpha \leftharpoondown \beta, y) = 1$ iff $t \leq_x^2 y$ $(f_x^2(\alpha,t)=1$ and $f_x^2(\beta,t)=0)$;
(7.1) $y\ \ X_x^1$: $f_x^1(\alpha\equiv\alpha,y)=1$;
(7.2) $y\ \ X_x^1$: if $f_x^1(\alpha\equiv\beta,y) = 1$, then $t\ \ \geq_x^1 y\ \ f_x^1(\alpha,t) = f_x^1(\beta,t)$;
(7.3) $y\ \ X_x^1$: if $f_x^1(\alpha\equiv\beta,y)=1$, then $f_x^1(\neg\alpha\equiv\neg\beta\ ,y)=1$;
(7.4) $y\ \ X_x^1$: if $f_x^1(\alpha\equiv\beta,y) = f_x^1(\gamma\equiv\delta,y)=1$, then $f_x^1((\alpha\S\gamma)\equiv(\beta\S\delta),y)=1$

for $\S\ \ \{\ ,\ \ ,\rightarrow,\leftrightarrow,\equiv\}$;

(8.1) $y\ \ X_x^2$: $f_x^2(\alpha\not\equiv\alpha,y)=0$;
(8.2) $y\ \ X_x^2$: if $f_x^2(\alpha\not\equiv\beta,y)=0$, then $t \leq_x^2 y\ f_x^2(\alpha,t) = f_x^2(\beta,t)$;
(8.3) $y\ \ X_x^2$: if $f_x^2(\alpha\not\equiv\beta,y)=0$, then $f_x^2(\ \ \alpha\not\equiv\ \ \beta y)= 0$;
(8.4) $y\ \ X_x^2$: if $f_x^2(\alpha\not\equiv\beta,y) = f_x^2(\gamma\not\equiv\delta,y) = 0$, then $f_x^2((\alpha\S\gamma)\not\equiv(\beta\S\delta),y) = 0$

for $\S\ \ \{\ ,\ \ ,\leftharpoondown,\rightleftharpoons,\not\equiv\}$.

We will say that a formula α is true under $\mathbf{f}_X$, an epistemic valuation for HBSCIN, if $f_x^i(\alpha,x) = 1$ for any $x\ \ \ X$ for any $f_x^i\ \ \mathbf{f}_X$ and $i=\{1,2\}$.

Proposition 4.2. HBSCIN is the set of all formulas true under every epistemic valuation for HBSCIN.

Proof. Similar to 4.1.

5. The intuitionistic possibility

Applying an approach mentioned in Introduction, this section deals with a certain HBSCIN-theory and its equivalent connection with some intuitionistic modal system.

On the base of HBSCIN, the axiomatic strengthenings have two complementary prospects, "positive" and "negative". We could extend an HBSCIN-theory with some special formulas, but the same result is obtained when we add weak (i.e. "Brouwerian") negations of these formulas to the complement of this theory . On the other hand, the extension of the complement of some theory by chosen formulas can be replaced by an extension of this theory by the ("Heyting") negation of these formulas. These natural facts follow from (e), (f), (g) in Section 3. Let us consider two sets of formulas

$A = \{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \quad C_{HBSCIN}(\quad)\}$,
$B = \{\neg(\alpha \not\equiv \beta) : \neg(\alpha \rightleftharpoons \beta) \quad C_{HBSCIN}(\quad)\}$.

The algebraic model for the extension

$$WIT = C_{HBSCIN}(A \quad B) \tag{11}$$

is a Heyting-Brouwer algebra with two additional binary operators $\circ$ and $\cdot$ satisfying, respectively ($\circ_1$)-($\circ_4$) and

($\cdot_1$) $a \cdot a = 0$
($\cdot_2$) $(\sim a \cdot \sim b) \leq (a \cdot b)$
($\cdot_4$) $((a \,\S\, c) \cdot (b \,\S\, d)) \leq ((a \cdot b) \quad (c \cdot d))$, $\quad \S \quad \{\cap, \quad , \leftharpoondown, \cdot\}$
($\cdot_4$) $(a \leftharpoondown b) \leq (a \cdot b)$

The HB-algebra with thus defined binary operators will be called topological Heyting-Brouwer algebra (tHBa). The reason for such name is rather clear. The relations between "white circle" operation and interior operator are established in Section 1. Now, it is sufficient to express the similar connections between "black circle" operation and the closure operator.

Using the translations

$$Ca = (a \cdot 0) \text{ and } (a \cdot b) = C(a \rightleftharpoons b) \tag{12}$$

for any *a* and *b* the elements of tHBa, it is easy to show that the conditions ($\cdot_1$)-($\cdot_4$) are equivalent to (C_1)-(C_4). Easy proof is left to the reader. As in the case of stHBa, the name "tHBa" will refer to HB-algebra with interior and closure operators and to HB-algebra with both circles operations satisfying the above mentioned conditions.

Proposition 5.1. The intuitionistic modal system IS4 is the set of all formulas which are true in each tHBa.

Proof. Obvious.

Notice that the well known connection holding in every pseudo-complemented lattice A with interior operator

$$Ia \cap Ib \leq I(a \cap b) \quad \text{iff} \quad I(a \to b) \leq Ia \to Ib \tag{13}$$

for any *a,b* A, have in tHBa its counterexample

$$C(a \quad b) \leq Ca \quad Cb \quad \text{iff} \quad Ca \leftharpoondown Cb \leq C(a \leftharpoondown b) \qquad (14)$$

for any *a* and *b* the elements of tHBa.

Let $L_{HBm} = (L_{HBm}, \neg, \ , \ , \ , \rightarrow, \leftharpoondown, \leftrightarrow, \rightleftharpoons, \Box, \Diamond)$. Now, as in the case of the classical base for modalities and identity we can formulate

Proposition 5.2. Let
1. f: $L_{HBSCIN} \longrightarrow L_{HBm}$, such that
$f(\alpha) = \alpha$, if "≡" and "≢" do not appear in α,
$f(\alpha \equiv \beta) = \Box(\alpha \leftrightarrow \beta)$,
$f(\alpha \not\equiv \beta) = \Diamond(\alpha \rightleftharpoons \beta)$.
2. g: $L_{HBm} \longrightarrow L_{HBSCIN}$,
such that
$g(\alpha) = \alpha$, if "□" and "◇" do not appear in α,
$g(\Box\alpha) = (\alpha \equiv (\alpha \rightarrow \alpha))$,
$g(\Diamond\alpha) = (\alpha \not\equiv (\alpha \leftharpoondown \alpha))$.
Then
(a) α WIT iff f(α) IS4, for any α L_{HBSCIN},
(b) α IS4 iff g(α) WIT, for any α L_{HBm}.

Proof. (a) Assume that α WIT. If in α neither "≡" nor "≢" appear, then α HB and so thesis. Let (α≡β) WIT. So, $h^{\equiv}(\alpha\equiv\beta)=1_{A}$ for any tHBa A and for any $h^{\equiv}$ Hom(L_{HBSCIN}, A). Now, let $f(\alpha\equiv\beta)=\Box(\alpha\leftrightarrow\beta)$ IS4. Then $h_0^{m}(\Box(\alpha\leftrightarrow\beta))\neq 1_{A0}$ and so $h_0^{m}(\alpha\leftrightarrow\beta)\neq 1_{A0}$ for some tHBa A_0 and for some h_0^{m} Hom(L_{HBm}, A). Let $h_{0|L\,HBSC}^{m}$ Hom(L_{HBSC}, $A_{0|HB}$), where L_{HBSC} is the language for Heyting-Brouwer Sentential Calculus and $A_{0|HB}$ is the HB-reduct of A_0. Let A_1 be an algebra $A_{o|HB}$ with additional operation ∘ equal to ↔. Let us define $h_1^{\equiv}$ Hom L_{HBSCIN}, A) such that $h_1^{\equiv}(\alpha)=h_{0|L}^{m}(\alpha)$, if ≡ does not appear in α. In other case $h_1^{\equiv}(\alpha\equiv\beta) = h_{0|L}^{m}(\alpha\leftrightarrow\beta)$. Since $h_1^{\equiv}(\alpha\equiv\beta)\neq\neg 1_{A1}$, thus contradiction. The proof for the nonidentity formulas is analogous.

Assume that f(α) IS4. If ≡ does not appear in α, then thesis. Let $f(\alpha\equiv\beta) = \Box(\alpha\leftrightarrow\beta)$ IS4, then $h^{m}(\Box(\alpha\leftrightarrow\beta)) = 1_{A}$ and so $h^{m}(\alpha\leftrightarrow\beta) = 1_{A}$ for any tHBa A and for any h^{m} Hom(L_{Hbm}, A). Now, assume that (α≡β)

WIT. By (13) $\alpha \leftrightarrow \beta$ is not a tautology of Heyting-Brouwer Sentential Calculus. It means that $h_0(\alpha \leftrightarrow \beta) \neq 1_{A0}$ for some HB-algebra A_0 and h_0 $\mathrm{Hom}(L_{\mathrm{HBSC}}, A)$. Now it is sufficient to extend this algebra to tHBa with a trivial identical function as an interior operator. Then $h_0^{m}(\alpha \leftrightarrow \beta) \neq 1_{A0}^{m}$ for tHBa A_0^{m} and h_0^{m} $\mathrm{Hom}(L_{\mathrm{HBm}}, A_0^{m})$ - a contradiction. The proof for the case of nonidentity is analogous.

The proof of (b) is similar to the just presented. □

Comment 1. The Heyting-Brouwer Sentential Calculus merits to be called a Full Intuitionistic Sentential Calculus. Its "Heyting" part being called Intuitionism enables us to decide what is true or not from the point of view of the future, only. But, of course, there are cases demanding the second perspective - the past.

Thus, for example, saying that α implies β we express the belief that it is impossible in the future that α will be true and β not. But, if we would like to say "α does not imply β", it means that we have already known that there was a situation in the past, when α was true and β not. It is easy to notice that such an expression is completely different from the negation of "α implies β" and from the case when "α implies β" does not hold. Similar remarks apply to the equivalence and the coequivalence ($\rightleftharpoons$).

An important conclusion follows for the connective of the intuitionistic possibility. The meaning of the intuitionistic necessity is unexpected. Saying that α is necessary we express that the truth of α is necessary (in the future) because α is at least from now equal to a tautology. In other words, α is not only true but it is necessarily true. When do we say that α is possible? To answer this question we need our experience! *We know that α is possible, because we have already known such a case (from the past) that α was different from the contradiction*. Simultaneously, it is not necessary in this case to know that α was true at any time before.

The case of intuitionistic S5. Both intuitionistic S4-modalities are introduced independently. It means that we can construct the Full Intuitionistic Sentential Calculus with, for example, the necessity of kind S4 and the possibility of kind S5.

Thus there is not the case to obtain S5-modalities forced only by additional conditions for necessity connective. We have to extend IS4 by two independent conditions: one for necessity and one for possibility.

There are different methods for defining the intuitionistic S5. We accept the Font's defining of the S5-necessity and adopt it to the case of the possibility (cf. [8]).

Definition 5.1.
(a) The tHBa *A* is a model for Heyting-Brouwer Sentential Calculus with necessity of kind S5, if B - the set of all open elements of *A* is a Boolean subalgebra of A.
(b) The tHBa *A* is a model for Heyting-Brouwer Sentential Calculus with possibility of kind S5, if T - the set of all closed elements of *A* is a Boolean subalgebra of A.

A natural criterion for algebraic models for HBSC with modalities of the kind S5 is given by

Proposition 5.3.
(a) The set of all open elements of tHBa *A* is a Boolean subalgebra of A, if for any a A

$$\mathrm{I}a \quad \mathrm{I}\neg\mathrm{I}a = 1. \qquad (15)$$

(b) The set of all closed elements of tHBa *A* is a Boolean subalgebra of A, if for any a A

$$\mathrm{C}a \cap \mathrm{C}{\sim}\mathrm{C}a = 0. \qquad (16)$$

Proof. Obvious. A HB-algebra *A* becomes Boolean, if for any a A $a \quad \neg a = 1$ (or equivalently $a \cap {\sim} a = 0$). The equality $\neg\mathrm{I}a = \mathrm{I}\neg\mathrm{I}a$ (${\sim}\mathrm{C}a = \mathrm{C}{\sim}\mathrm{C}a$) means that the set B (T) is closed under negation (weak negation). Thus the conditions (15) and (16) imply the theses. □

The both parts of the definition 5.1 can equivalently be strengthened to the conditions according to which B and T are two-element sets.

Proposition 5.4. In every tHBa *A* the condition (I_5) is equivalent to ($\circ_5$), and the condition (C_5) is equivalent to ($\cdot_5$).

(I_5) $\mathrm{B} = \{\mathrm{I}a: a \quad \mathrm{A}\} = \{0,1\}$
(C_5) $\mathrm{T} = \{\mathrm{C}a: a \quad \mathrm{A}\} = \{0,1\}$

$$(\circ_5)\ a \circ b = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$

$$(\cdot_5)\ a \cdot b = \begin{cases} 1, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}$$

Proof. Implications: $(\circ_5) \dashrightarrow (I_5)$ and $(\cdot_5) \dashrightarrow (C_5)$ follow directly from the translations Ia = $(a\circ 1)$ and Ca = $(a\cdot 0)$. Assume (I_5). If $a=b$, then $(a\circ b) = I(a \leftrightarrow b) = I1 = 1$. When $a \neq b$, then $(a\circ b) \leq (a \leftrightarrow b) < 1$, so $(a\circ b) = 0$. Now assume (C_5). If $a=b$, then $(a\cdot b) = C(a \rightleftharpoons b) = C0 = 0$. When $a \neq b$, then $(a\cdot b) \geq (a \rightleftharpoons b) > 0$, so $(a\cdot b) = 1$. □

Now, let us repeat the Greniewski's axiom and add its "Brouwerian" counterpart

(A_5) $((\alpha \equiv \beta) \equiv 0) \quad ((\alpha \equiv \beta) \equiv 1)$,
(B_5) $\neg((\alpha \not\equiv \beta) \not\equiv 0) \quad \neg((\alpha \not\equiv \beta) \not\equiv 1)$.

Proposition 5.5.
(a) The tHBa A satisfying the condition $(\circ_5)$ is a model for WIT+(A_5).
(b) The tHBa A satisfying the condition $(\cdot_5)$ is a model for WIT+(B_5).

Proof. Obvious.

At the beginning of Section 2 some extensions of the intuitionistic system of the kind S4 by Font were discussed. It is easy to see, that in our case it is possible to consider all the systems mentioned, remembering that axioms for necessity connective do not "work" for the connective of possibility. Moreover, our strengthening can be developed in "connected" form or free from the axioms: $(\Box\Diamond_1)$ and $(\Box\Diamond_2)$.

The Example 4 (see Appendix) professes independence of the conditions

$$C(a \to b) \leq Ia \to Cb \quad \text{and} \quad Ca \to Ib \leq I(a \to b)$$

from (15) and (16).
The two following systems

$$\text{WIH} = \text{WIT} + \{A_5, B_5\}$$

$$\text{IS5} = \text{IS4} + \{\Box\alpha \quad \Box\neg\Box\alpha,\ \neg(\Diamond\alpha \quad \Diamond\ \Diamond\alpha)\}$$

are equivalent in the sense explained below.

Proposition 5.6. Let f and g be the functions defined in the Proposition 5.2. Then

(a) α WIH iff $f(\alpha)$ IS5, for any α L_{HBSCIN},
(b) α IS5 iff $g(\alpha)$ WIH, for any α L_{HBm}.

Proof. Analogous to the one for proposition 5.2.

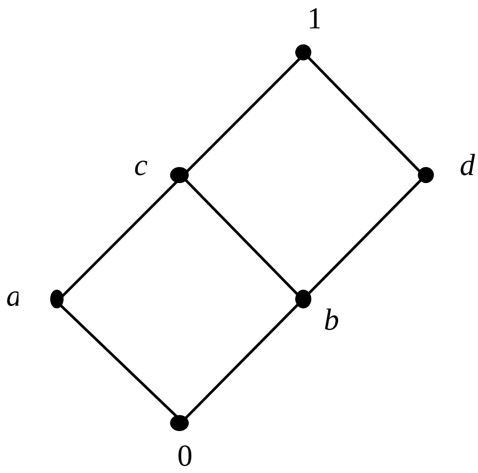
Comment 2. The "Brouwerian" nature of the connective of possibility is emphasized by the fact that an extension axiom for the intuitionistic possibility of S5 kind is as follows:

$$\neg(\Diamond\alpha \quad \Diamond\ \Diamond\alpha). \tag{17}$$

It is impossible to replace this formula by the any other containing "Heyting's" connectives only. Therefore, there are many different intuitionistic modal systems of S5 built on the "Heyting's" part of the language for Heyting-Brouwer Sentential Calculus. Obviously, none of them would satisfy the "natural" Definition 5.1(b).

6. Appendix

Example 1. Assume the stHBa A given on the diagram. Let B = {I*a*: *a*   A} = {0, *a*, 1}. Then, T = {C*a* : *a*   A} = {0, *d*, 1}. By an easy calculation one can check all conditions from 3.2 and 3.3. A is not, however, weakly monadic. Indeed, $\neg Cb = \neg d = a \neq 1 = Ca = C\neg Cb$.



Example 2. For any α,β L_{HB} $\alpha \to \beta$ T iff $\alpha \leftharpoondown \beta$ $\overline{T}$ (see [15]).

Assume that for some α,β L_{HBSCIN} and some HBSCIN-theory T $\alpha \approx_T \beta$. Thus $\alpha \not\approx \beta$ $\overline{T}$ and due to (B_3) $((\beta \not\approx \alpha) \not\approx (\beta \not\approx \beta)) \leftharpoondown ((\beta \not\approx \beta)$ $(\alpha \not\approx \beta))$ $\overline{T}$. Because of $\beta \not\approx \beta$ $\overline{T}$ and from the assumption: $(\beta \not\approx \beta)$ $(\alpha \not\approx \beta)$ $\overline{T}$. Thus $(\beta \not\approx \alpha) \not\approx (\beta \not\approx \beta)$ $\overline{T}$. By axiom (B_3) $(\beta \not\approx \alpha) \leftharpoondown (\beta \not\approx \beta)$ $\overline{T}$. Applying again the rule dual to the Modus Ponens we get $(\beta \not\approx \alpha)$ $\overline{T}$, so $\beta \approx_T \alpha$.

For more about the relations between Heyting-Brouwer-theories and their complements see [15].

Example 3. We show that the formula

$$\neg(\alpha \not\approx \beta) \to (\alpha \equiv \beta)$$

is not a tautology of HBSCIN. Assume that for some HBSCIN-model M, for some h Hom(L_{HBSCIN},A) (A is the base of the construction of M) for some x X and k_x,
k_x o $h(\neg(\alpha \not\approx \beta) \to (\alpha \equiv \beta))$ D_x. Thus
$k_x(\neg(h(\alpha) \cdot h(\beta)) \to (h(\alpha) \circ h(\beta)))$ D_x. By 4.1(5)
$x_1 \geq_X x$ $(k_{x1}(\neg(h(\alpha) \cdot h(\beta)))$ D_{x1} and $k_{x1}(h(\alpha) \circ h(\beta))$ D_{x1}). By (11)
$x_1 \geq_X x$ ($y \geq_X x_1$ $k_y(h(\alpha) \cdot h(\beta))$ D_y and $x_2 \geq_X x_1$ $k_{x2}(h(\alpha)) \neq k_{x2}$ $(h(\beta)))$.

By (10) we know that t Y k_t (h(α)· (β)) D_t and so
t Y $z \leq_Y t$ $k_z(h(\alpha)) = k_z(h(\beta))$.
Thus for x_2 X there exists t_1 Y such that for any δ L_{HBSCIN}
$k_{x2}(h(\delta))$ D_{x2} iff $k_{t1}(h(\delta))$ D_{t1} and moreover $k_{t1}(h(\alpha)) = k_{t1}(h(\beta))$.
Although $k_{x2}(h(\alpha)) \neq k_{x2}(h(\beta))$ there is no contradiction, because the equivalence classes given by k_{x2} and k_{t1} can be different.

The example above includes an element showing that the connectives ≡ and ≋ are entirely independent. Namely, the division between two "worlds" - one is determined by the order $\leq_X$ and canonical homomorphisms with the relations of the first kind, and the other is established by the order $\leq_Y$ and canonical homomorphisms connected with the relations of the second kind.

Example 4. Let the tHBa be given by the diagram from Example 1 and let B = {0, a, d, 1}, T = {0, 1}. An easy verification shows that thus given interior and closure operators satisfy the conditions (15) and (16). But, simultaneously

$$C(d \to 0) = Ca = 1 > a = d \to 0 = Id \to C0.$$

Now let interior and closure operators satisfying (15) and (16) be given the sets B = {0, 1} and T = {0, a, d, 1} on the same tHBa. Then

$$Cd \to I0 = d \to 0 = a > 0 = Ia = I(d \to 0).$$

This falsifies the inequalities interpreting axioms (□◇$_1$) and (□◇$_2$).

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