Logica Trianguli, 3, 1999, 55-83

A SURVEY OF NATURAL DEDUCTION SYSTEMS FOR MODAL LOGICS

Andrzej INDRZEJCZAK

Abstract

The paper contains an exposition of standard ND-formalizations for modal logics. For the sake of simplicity, it is limited to *propositional monomodal logics* because focus is on methods not on logics. Some of the discussed approaches, however, may be easily extended to first order modal logics of different sorts (see [16]) or to *multimodal logics* (e.g. for temporal logics see [17]). Natural Deduction is understood in the strict sense, explained below; neither *Gentzen Sequent Calculus,* nor *Tableau Systems* belong to that group. Moreover, some ND-systems with generalized apparatus, like [2] or [3] are also omitted in this survey. There are four basic approaches to modality via ND-formalization, which are compared and evaluated with respect to their generality: modalization of assumptions, modalization of reiteration rule, modalization of rules, application of modal assumptions. We focus on practical matters, especially on the applicability of discussed approaches, the only exception is the discussion of independent rules for necessity and possibility in the last section.

1. Preliminaries

1.1. Logics

We will use standard language <¬, →, , , □>. For the most part we will use the possibility operator ◇ as a defined shortcut for ¬□¬. The exception is the last section where ◇ is rather used as basic. φ, ψ refer to any formula, Γ, Δ, Π, Σ to any set of formulae, usually finite. ¬Γ, □Γ, ◇Γ mean, respectively, {¬φ:φ Γ}, {□φ:φ Γ}, {◇φ:φ Γ}, and Γ, Γ a conjunction/disjunction of all formulae in Γ. □-formula is any formula prefixed with □ or ¬◇, and ◇-formula by ◇, or ¬□; M-formula is any □- or ◇-formula.

Let us remind that *propositional monomodal regular logic* is any logic **L** which satisfies the following conditions:

L contains the set of theses of **CPC** (Classical Propositional Calculus)

Axiom K) belongs to **L** i.e. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ **L**

L is closed with respect to MP) and C) i.e.

if $\{\phi \to \psi, \phi\}$ **L** , then ψ **L,** and

if $\psi \to \phi$ **L** , then $\Box\psi \to \Box\phi$ **L**

The weakest regular logic is usually called **C**, if we replace C) by a stronger rule GR) (Gödel's Rule) i.e.

if ϕ **L** , then $\Box\phi$ **L**

we obtain the weakest *normal logic*, usually called **K**.

The extensions of **C** (or **K**) are obtained by addition of extra axioms. The most popular that will serve as examples, are displayed below:

D) $\Box\phi \to \Diamond\phi$

T) $\Box\phi \to \phi$

4) $\Box\phi \to \Box\Box\phi$

B) $\phi \to \Box\Diamond\phi$

E) $\Diamond\phi \to \Box\Diamond\phi$

G) $\Box(\Box\phi \to \phi) \to \Box\phi$

Since logics are build by addition of axioms from this list to **C** (or **K**) we will give them names by concatenation of names of axioms e.g. **K** + B) + 4) will be simply noted as **KB4**. But one should notice that some of them have traditional names which we shall preserve. These are:

D = **KD**

T = **KT**

G = **K4G**

B = **KTB**

S4 = **KT4**

S5 = **KT4E** = **KT4B** = **KTE**

Although we do not need to remind Kripke semantic for modal logic, by obvious correspondence to semantic conditions we will also use some names for distinctive groups of logics. All logics containing T) will be called *reflexive* (and irreflexive otherwise), containing B), *symmetric* (oth. nonsymmetric), containing 4), *transitive* (oth. nontransitive).

Instead of saying that ϕ **L,** we may put $\vdash_{\mathbf{L}}$ (subscript usually will be omitted when not necessary), which means that ϕ is a thesis of **L** (has a proof in **L**). Provability is characterised as follows $\Gamma \vdash \phi$ iff $\vdash \ \Gamma \to \phi$. Perhaps it is not the most popular definition of provability in axiom systems, because it is assumed that C) and GR) was not used to elements of Γ, but it is the most natural definition for ND-systems because it involves *deduction theorem* in ordinary shape; such relation is sometimes called *truth consequence relation* (see e.g. [2]). Items of the form $\Gamma \vdash \phi$ (with Γ possibly empty) will be called *sequents* and they clearly correspond to elementary inferences *derivable* in the respective logic. More general notion which is of great interest is the notion of a *sequent rule* of the form: if S1,…, Sn, then Sn+1, where S refers to any sequent. The example of a sequent rule is in fact GR), which should be written: if $\vdash \phi$, then $\vdash \Box\phi$. The other important sequent rule is the rule:

$\Box$) if $\Gamma \vdash \phi$, then $\Box\Gamma \vdash \Box\phi$

which is satisfied, or better *admissible*, in all normal logics and also in all regular ones but with proviso that $\Gamma \neq$. Clearly each sequent corresponds to some sequent rule in the manner expressed by the lemma:

Lemma 1: *if* $\{\phi 1,\ldots,\phi n\} \vdash_{\mathbf{L}} \psi$, *then if* $\Gamma 1 \vdash_{\mathbf{L}} \phi 1, \ldots, \Gamma n \vdash_{\mathbf{L}} \phi n$, *then* $\Gamma 1,\ldots,\Gamma n \vdash_{\mathbf{L}} \psi$

which means that each rule derivable in **L** is also admissible in **L**; proof is obvious, by n applications of the cut-rule. In the following we will use the convention that if the sequent has some name we will use the same name but with "S" in superscript, for corresponding sequent rule e.g. E) will be a name for a sequent expressing elimination of conjunction and E^S) will be a name for suitable sequent rule. Obviously all theses, and more generally, sequents, derivable in some logic are derivable in all extensions of that logic. This is not in general true with respect to sequent rules; something admissible in a logic is not necessarily admissible in some extension but, fortunately, all sequent rules presented in this paper behave very nice in this respect, so we will simply state them as admissible for some logic and all its extensions that we consider.

1.2. Natural Deduction

It seems that no precise definition of ND-systems has been offered so far. The term is often used in a very broad sense so that it covers almost everything which is not an axiom system; here it is taken in a narrow sense. ND-system is meant as the one in which there are some rules for entering assumptions into a proof and also for discharging them. There are no (or, at least, very limited set of) axioms, because their role is taken over by the set of rules for introduction and elimination of constants; it means that elementary inferences instead of formulae are taken as primitive. Genuine ND-system admits a lot of freedom in proof construction, both direct and indirect proofs are possible, one can build more complex formulae, or decompose them, as respective introduction/elimination rules allow. This flexibility is significant for ND-system in contrast to *Gentzen Sequent Calculus*, where proofs are cumulative, due to the application of only introduction rules, or to

Tableau Systems, where proofs are always indirect with only elimination rules being applied.

Many existing systems satisfy this characteristics but differ in many other respects. The most important features, for us, are: proof construction and basic items, for which rules of the system are defined. We will consider systems where basic items may be formulae or sequents, accordingly we divide ND-systems on F-, and S-systems. Proof construction may be *linear* or in a *tree form*, hence we can divide ND-systems on L-, and T-systems. The latter distinction is not important for S-systems, but it really matters in F-systems, because if a proof is linear we may use the same formula many times, hence we must have some devices to cancel the part of a proof which is in the scope of an assumption already discharged. Otherwise we could prove anything. This is not possible in T-proofs because we are operating not on formulae but on their occurrences; premises of a rule must always be displayed directly over the conclusion, so we cannot use something which depends on discharged assumption. Further in the paper we will rather concentrate on L-systems because they are of practical interests. F-L-systems will be called systems in *Jaśkowski format* (see [20] for the first system of this sort), S-systems will be called systems in *Gentzen format* (see [13] for the first S-T-system, earlier in [12] Gentzen devised the first F-T-system; for more about these matters consult [19]).

Any type of a *Deductive system*, including ND-systems, may be characterised by listing of all primitive sequents and sequent rules of this system. These two nonempty sets build up the *content* of the system, a theoretical level on which we can compare different systems. The content of a basic F-system and S-system for **CPC** is listed below:

I) $\varphi, \psi \vdash \varphi \quad \psi$
E) $\varphi \quad \psi \vdash \varphi \qquad \varphi \quad \psi \vdash \psi$
I) $\varphi \vdash \varphi \quad \psi \qquad \psi \vdash \varphi \quad \psi$
E) $\neg\varphi, \varphi \quad \psi \vdash \psi$
$\rightarrow$E) $\varphi, \varphi \rightarrow \psi \vdash \psi$
$\rightarrow$I) if $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash \varphi \rightarrow \psi$
$\neg$E) if $\Gamma, \neg\varphi \vdash$, then $\Gamma \vdash \varphi$
I) $\varphi, \neg\varphi \vdash$ E) $\vdash \varphi$

A) $\varphi \vdash \varphi$
W) if $\Gamma \vdash \psi$, then $\varphi, \Gamma \vdash \psi$
I^S) if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$, then $\Gamma, \Delta \vdash \varphi \quad \psi$
E^S) if $\Gamma \vdash \varphi \quad \psi$, then $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$
I^S) if $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi \quad \psi$ or $\Gamma \vdash \psi \quad \varphi$

E^S) if $\Gamma \vdash \phi \quad \psi$ and $\Delta \vdash \neg\phi$, then $\Gamma, \Delta \vdash \psi$

$\rightarrow E^S$) if $\Gamma \vdash \phi$ and $\Delta \vdash \phi \rightarrow \psi$, then $\Gamma, \Delta \vdash \psi$

$\rightarrow I$) if $\Gamma, \phi \vdash \psi$, then $\Gamma \vdash \phi \rightarrow \psi$

$\neg E$) if $\Gamma, \neg\phi \vdash$, then $\Gamma \vdash \phi$

I^S) if $\Gamma \vdash \phi$ and $\Delta \vdash \neg\phi$, then $\Gamma, \Delta \vdash$

The content of a system is a base for soundness proofs for respective formalizations. One can easily check that above rules are sound in **CPC** if we replace $\vdash$ by $\models$, the sign of entailment. All the rules defined further for modal logics may be easily checked in this way. The content is not yet the full characteristics of a system. There is also a level of *representation* which refers to practice. It is a set of instructions of how to build a proof, apply rules and so on. In case of Jaśkowski format there is also a need for additional devices to show that some part of a proof is not in force, because its assumption is discharged. There are many known representations, we may call them variants, in what follows we will choose a variant due to Kalish and Montague [21] as a representation of Jaśkowski format and a variant due to Suppes [26] as a representation of Gentzen format. There is no place for detailed exposition, hence we will restrict ourselves only to the most important properties of both variants.

In general in ND-system, on the level of representation we have at least two kinds of rules: *rules of inference* and *rules of proof construction* (or shortly proof rules). This division is independent of that for sequents and sequent rules on the level of content; although in F-systems sequents are inference rules and sequent rules are proof rules, in S-system some sequent rules are rules of inference and some proof rules. Moreover, we have usually additional types of rules as we shall see in the system of Kalish and Montague, from now on called **K&M**. The special feature of this system is that we have also two different kinds of lines in a proof:

a) *usable lines*, that contain premises, assumptions and conclusions of applied rules;

b) *show lines* displaying formulae that one attempts to prove; these are of the form “Show: ϕ”. Show lines are present in a proof only temporary, when we succeeded in a proof of ϕ by some method its line changes into usable line. Accordingly, we will call all formulae displayed in show lines as *show-formulae* and all other as *usable-formulae*.

In **K&M** a proof of ϕ starts with a show line for this formula as a main goal and during the proof construction one can enter as many show lines as wanted, thus dividing the process of proving into the realization of many simpler subgoals. We can say that every show line opens k-degree derivation (k>0) and if the goal of this derivation is realized its show-formula becomes usable-formula of k-1 degree (i.e. outer) derivation. In such a case we cancel the prefix "Show" and close the whole derivation of k-degree in a box. Proof is finished if the first show line is cancelled and the whole proof below it (a derivation of 1-degree) is boxed. There are three proof rules that allow for closing a derivation:

[Dir]: Let ϕ be a show-formula of k-degree derivation, then we can close this derivation provided ϕ has appeared as a usable-formula in this derivation.

[Cond]: Let $\psi\rightarrow\phi$ be a show-formula of k-degree derivation, then we can close this derivation provided ϕ has appeared as a usable-formula in this derivation.

[Red]: Whenever has appeared as a usable-formula in this derivation (in this case the shape of a show-formula is of no importance).

Schematically:

```
          :                                    :
i-j       Γ                          i-j       Γ
          :                                    :
k         ~~Show~~:ϕ    [Dir,n]      k         ~~Show~~:ϕ→ψ    [Cond,o]
l-m      | Γ'‾ |        R)i-j         k+1     | (ϕ) |          assum.
         | :   |                      m-n     | Γ'  |          R)i-j
         | :   |                              | :   |
n        | ϕ   |        i-m           o       | ψ   |          k+1, m-n
```

```
             :
      i-j    Γ
             :
       k     ~~Show~~:ϕ    [Red,o]
      k+1   | (¬ϕ) |       assum.
      m-n   |  Γ'  |       R)i-j
            |  :   |
       o    |      |       k+1, m-n
```

Remark 1. The formulation of rules forbids closing of a derivation of k-degree if there are some show lines in this derivation because each of them enters a subderivation of a degree>k, hence if the formula required to closing a derivation is in their scope it is not an usable formula of k-degree derivation. Hence each initiated subgoal must be realized before we finish the whole proof.

Assumptions in **K&M** may be entered optionally, so on the schemas they are in brackets. They are always the first lines of a derivation:
a) if the last show-formula is $\varphi \rightarrow \psi$, then we may add φ as a *conditional assumption* of this derivation;
b) if the last show-formula is φ, then we may add $\neg\varphi$ as an *indirect assumption* of this derivation.

Γ' on the above schemas displays the usable-formulae from Γ that were transported from outer derivation of k-1 degree to k-degree derivation by reiteration rule R). In case of **CPC** this rule simply transports formulae without any restriction from open derivation of some degree to each open derivation of higher degree. Of course, the reverse direction and any moves from closed derivation are forbidden. The very importance of this rule in the context of modal logics will become evident in the next section.

All inference rules of **K&M** are simply sequents listed in the content of any F-system.

In S-system we have for each formula in succedent a record of its assumptions in the antecedent of a sequent, hence there is no need for some bookkeeping devices blocking forbidden deductions. One should notice that all the introduction and elimination rules operate not really on sequents but on their succedents. The only operation being made on antecedents is addition and subtraction of formulae. It allows for some simplification which is used in Suppes variant; we can get rid with the rewriting of all sequents, replacing formulae in antecedents by the numbers of lines where they have appeared for the first time as assumptions (justified by rule A)). In the next section we will see an example of a proof in Suppes' variant for Gentzen format.

Remark 2. One should notice that our characterisation is limited to something which may be called a standard ND-system in the spirit of the fathers of these systems: Jaśkowski and Gentzen. In what follows we compare only such ND-formalizations of modal logic that are based on this standard. But there are many interesting generalizations of ND-systems that sometimes allow for some extra results see e.g. [2], where different consequence relations are manipulated in one system, or [3] for ND-system defined on labelled formulae. But all these non-standard approaches are beyond this study which is sufficiently long.

2. Necessity based approaches

2.1. Modalized assumptions

The first approach to the extension of ND-techniques to modal logics was based on the concept of modalized assumptions. It is due to Curry [8] for **S4**, [5] contains system for **S4** and **S5**; see also [23] and [7]. The idea is that the application of $\Box$I) to formula is dependent of the shape of assumptions of the formula in question. It should be limited to cases where the set of assumptions is empty or consists only of modalized assumptions. In the first case it is simply an application of GR), in the second we must check whether all the assumptions satisfy suitable conditions.

The advantage of this approach is its independence of the basic format; Curry and Prawitz [23] have used Gentzen T-F-format, Borkowski & Słupecki [5] Jaśkowski format. In fact, any variants of Gentzen format seem to be better prepared to this task because all actual assumptions of each formula are displayed. In Jaśkowski format a formula may be put in the scope of assumptions, on which it is not in fact dependent, hence the control whether $\Box$I) may be applied is harder. This is the reason why in this section we will use Gentzen format in Suppes variant for examples.

The serious drawback of this system is the lack of generality. It is not incidental that it is devised only for **S4** and **S5**; it is certainly not obvious how to extend this approach to other logics.

In basic system modalized formulae are defined for **S4** as any $\Box$-formulae, and for **S5**, additionally as any $\Diamond$-formulae. Hence we have the following rules $\Box$I):

if $\Box\Gamma \vdash \phi$, then $\Box\Gamma \vdash \Box\phi$, for **S4** and

if $\Diamond\Delta, \Box\Gamma \vdash \phi$, then $\Diamond\Delta, \Box\Gamma \vdash \Box\phi$, for **S5**

and the same rule $\Box E^S$) for both logics:

if $\Gamma \vdash \Box\phi$, then $\Gamma \vdash \phi$

The above definition of modalized formulae gives simplified account of rules but, unfortunately, forces us to construct unnecessarily long and complicated proofs; here is an example:

{1}	1	□φ □ψ	A)
{1}	2	□φ	E^S),1
{1}	3	□ψ	E^S),1
{4}	4	□φ	A)
{4}	5	φ	□E^S),4
{6}	6	□ψ	A)
{6}	7	ψ	□E^S),6
{4,6}	8	φ ψ	I^S),5,7
{4,6}	9	□(φ ψ)	□I),8
{4}	10	□ψ→□(φ ψ)	→I),9
	11	□φ→(□ψ→□(φ ψ))	→I),10
{1}	12	□ψ→□(φ ψ)	→E^S),2,11
{1}	13	□(φ ψ)	→E^S),3,12
	14	(□φ □ψ)→□(φ ψ)	→I),13

The problem illustrated here is not only of practical nature, it has also some important theoretical aspect which we describe briefly. Prawitz [23] has proved the so called *normalization theorem* for many logics in ND-formalization. It shows that any ND-proof may be transformed into some normal form, which is the most economical form of the proof. In a proof in normal form first one applies elimination rules to assumptions, and then introduction rules. Hence, formulae are first decomposed (simplified), then mixed again in a suitable order and a theorem is derived in a very direct way, without unnecessary transformations. Prawitz' theorem is for ND the counterpart of a famous Gentzen's Cut-elimination theorem. The characteristic feature of proofs that are not in normal form is the presence of the so called maximal formulae which are conclusions of some introduction rules, and then they are used as premises for elimination rules, applied to the same constant. It means that compounding some formula precedes decomposing, which makes proof longer and more complicated. In the above example maximal formulae are present in lines 10 (12) and 11. Unfortunately, not all proofs in **S4** and **S5** may be transformed into normal form in this ND-system. Prawitz proposed the second variant where the definition of modalized formulae (MF) is more liberal; for **S4**:

a) and □φ, for any φ, belong to MF;
b) if ψ and φ belong to MF, then ψ φ and ψ φ also belong to MF.

For **S5** it is necessary to add in a) $\Diamond\varphi$, and in b) $\psi \rightarrow \varphi$; equivalently, and simpler, one can define as MF for **S5** any formula where each variable is in the scope of a modal functor. Admissibility of these rules is justified by the following:

Lemma 2. *If* φ *is modalized with respect to* **S4** (**S5**), *then* $\varphi \rightarrow \Box\varphi$ **S4** (**S5**).

Proof, by induction on the length of φ , in [23].

The latter definition allows to construct a proof for our example in normal form because the assumption in line 1 is a modalized formula. Unfortunately, this variant does not satisfy normalization theorem either. Notice that if in **S4**(**S5**) $\Gamma \vdash \Box\varphi$, where Γ contains only modalized formulae, then also $\psi, \Gamma \vdash \Box\varphi$, where ψ is not MF, but then $\{\psi\}$ $\Gamma \vdash \Box\varphi$; obviously $\{\psi\}$ Γ is not modalized according to our definition, hence the proof of $\Box\varphi$ on its basis requires some maximal formulae again.

Remark 3. Prawitz [23] presented also ND-systems for **S4** and **S5** which satisfy normalization theorem but these systems are of rather different sort. It is admissible in them to apply $\Box$I) to formula based on any assumption, on condition that there are some modalized formulae in the proof connecting these assumptions and a formula in question. In fact, it is rather a variant of Fitch's approach described in the next paragraph. It is also of no practical importance because it only shows what to check in a completed proof, not how to construct it.

2.2. Modalization of Reiteration Rule

Fitch [9] and [10] is the author of the next, and probably the most popular approach. His system is not universal, in the sense that it is not suitable for Gentzen format; some variant of Jaśkowski format is presupposed because it is essential in this approach to separate parts of the proof. We will present it in the **K&M** representation. In ND-system for **CPC** the reiteration rule R) is not specially limited; one can put any formula from open derivation of k-degree to open derivation of any higher degree. In case of modal logic, R) should be limited because a special category of derivations is added, namely *strict derivations*. If subderivation of k-degree is strict, then only special sort of formulae from outer derivation may be transported. The logic in question decides what kind of formulae is admissible. The idea is that one can obtain ND-formalizations for different logics only by modelling the set of suitable formulae, keeping all the inference and proof rules invariant.

Despite the limitation to Jaśkowski format, Fitch's solution has one unquestionable advantage - the scope: [9] contains only ND-system for **T** and **S4**, [10] has counterparts for some deontic logics, [25] extends this formalization further, finally in [11] one can find uniform formalization for many regular and normal logics. It is generalized even further, for many *relevance logics* [1], *conditional logics* [27] and *temporal logics* [17]. These are only some examples of application of Fitch's idea.

The rule $\Box$E) is a sequent $\Box\varphi \vdash \varphi$, and it is added only to reflexive logics. To the rules for closing a derivation we must add the rule:

[Nec]: Let $\Box\varphi$ be a show-formula of k-degree derivation, then we can close this derivation provided φ has appeared as a usable-formula in this derivation and there is no indirect assumption $\neg\Box\varphi$ under the show-line.

We have to modify also reiteration rule:

R): Let Γ be the set of usable-formulae of k-degree derivation and φ belongs to Γ, then φ may be added to k+1-degree derivation if either
a) the show-formula opening k+1-derivation is not of the form $\Box\psi$,
or
b) the first usable-formula of this derivation is an indirect assumption.
If neither a) nor b) is satisfied, then φ may be added provided φ belongs to Γ^*, where $\Gamma^* \quad$ Subfor$(\Diamond\Gamma)$.

Γ^* is defined for some logics as follows:

a) $\{\varphi : \Box\varphi \quad \Gamma\}$, for all logics
b) $\{\Box\varphi : \Box\varphi \quad \Gamma\}$, for all transitive logics
c) $\{\Diamond\varphi : \varphi \quad \Gamma\}$, for all symmetric logics
d) $\{\Diamond\varphi : \Diamond\varphi \quad \Gamma\}$, for symmetric and reflexive logics
the sum of Γ^* a) and b) for transitive and irreflexive logics like **K4** and its irreflexive extensions (like **G**)
the sum of Γ^* b) and d) in case of **S5**

The rule [Nec] is a counterpart of $\Box$I) rule in original Fitch system. In the system representation [Nec] and R) together are the counterpart of the rule $\Box$I) from the content level:

if $\Gamma^* \vdash \phi$, then $\Gamma \vdash \Box\phi$.

This is illustrated by the following scheme:

	:	
i-j	Γ	
	:	
k	~~Show~~:$\Box\phi$	[Nec,n]
l-m	Γ^*	R)i-j
	:	
n	ϕ	i-m

Informally this rule can be read: if ϕ is derivable in derivation of k+1-degree based on assumptions in Γ^* Subfor($\Diamond\Gamma$), then $\Box\phi$ has a proof of k-degree based on Γ. Admissibility of such a rule is easily established for each logic in question with the help of our table for Γ^*. In case of regular logics one should add to [Nec] a condition to the effect that subderivation of k+1-degree may be closed by this rule if at least one usable-formula of k-degree proof is a $\Box$-formula. Fitting [11] in his formalization for regular logics used a different but equivalent solution; instead of [Nec] he proposed a rule based on the principle:

if $\Gamma^* \vdash \phi \quad \psi$, then $\Gamma \vdash \Diamond\phi \quad \Box\psi$.

Remark 4. Rules [Nec] and R) may be significantly simplified. First, one can eliminate from **K&M** both [Red] and the rule for entering indirect assumptions, because these rules are not necessary in the system, where we have two additional inference rules:

MT) $\phi \rightarrow \psi, \neg\psi \vdash \neg\phi$ and
DN) $\phi \vdash \neg\neg\phi$, $\neg\neg\phi \vdash \phi$

The lack of [Red] and indirect assumptions forces us only to the more elaborate proof-constructions. In so modified system both [Nec] and R) may be formulated as follows:

[Nec]’ Let $\Box\phi$ be a show-formula of k-degree derivation and has ϕ appeared as a usable-formula in it, then we can close this derivation, provided all its usable-formulae justified by R) belong to Γ^*.

R)’ Let Γ be the set of usable-formulae of k-degree derivation and Γ^* Subfor($\Diamond\Gamma$), then we may add to k+1 derivation either:
a) ϕ Γ^* , if show-formula of this derivation is $\Box$-formula,
or
b) ϕ Γ .

Even if [Red] and indirect assumptions are kept intact, both rules can be simplified in case we formalize reflexive logics, so we can use [Nec]’ and

R)” Let Γ be the set of usable-formulae of k-degree derivation and Γ^* Subfor($\Diamond\Gamma$), then we may add to k+1 derivation either a) ϕ Γ^* or b) ϕ Γ.

This simplification is due to the fact that although in case of [Nec] R) must be limited to Γ^*, then in case of other rules of closing a derivation both Γ and Γ^* are admissible in reflexive logics, because formulae from Γ^* may be inferred by the application of admissible inference rules. Also the condition that indirect assumption should not be present in the derivation to be closed by [Nec] is not needed because in **T** and its extensions treated here one can prove:

if $\Gamma^*, \neg\Box\phi \vdash \phi$, then $\Gamma \vdash \Box\phi$.

Our primary formulation, despite the complications in the definition of R) has one serious advantage. All the necessary limitations are put as conditions to be satisfied before we apply the rule, hence we do not need to control finished proof if there are some mistakes. Simpler formulations, often found in literature, usually require some control of correctness after the proof is completed.

Remark 5. The drawback of this formalization is the lack of any rule $\Box$E) for logics weaker than **T**; in case of **D** one can use in this role a sequent D) $\Box\phi \vdash \Diamond\phi$, but it is an ad hoc solution. Providing the definition of Γ^* for **K**, it is possible to define R) for all nonstrict derivation once, and for [Nec] to define R) for each logic separately as a kind of

$\Box$E). This solution has also some disadvantages; e.g. in **K4**, $\Box$ is not only eliminated but also the whole $\Box$-formula is put again.

A different solution is possible, at least for some irreflexive logics, if we consider some obvious variants of $\Box$E). For example:

$\Box E^{\Box}$)	$\Box\phi \vdash \phi$,	for any $\Box$-formula ϕ
$\Box E^{\Diamond}$)	$\Box\phi \vdash \phi$,	for any $\Diamond$-formula ϕ
$\Box E^{\Box\Diamond}$)	$\Box\phi \vdash \phi$,	for any M-formula ϕ

$\Box E^{\Box}$) may be added to **KE** and its extensions, $\Box E^{\Diamond}$) to **K4D**, which implies that $\Box E^{\Box\Diamond}$) is a suitable rule for **K4ED**. In this way, at least some irreflexive logics have $\Box$E) of some sort, but it should be noticed that all these rules are in fact derivable in basic formalization, what is more, they cannot replace the rule D) in **KD4** or **KDE**. The exception is **KD4E**, a very important epistemic logic; instead to replace in **DN-S5** $\Box$E) by D), one can use $\Box E^{\Diamond}$), which is simpler and more natural. Proofs of K), 4), 5) run like in **DN-S5**, it is enough to show that D) is also provable; for shortcut we will use secondary rule R5) $\Diamond\Box\phi \vdash \Box\phi$, which is obviously derivable either.

1	~~Show~~: $\Box\phi \rightarrow \Diamond\phi$	[Cond,12]
2	$\Box\phi$	assumption
3	~~Show~~: $\Box\Diamond\phi$	[Red,4,8]
4	$\neg\Box\Diamond\phi$	assumption
5	$\Diamond\Box\neg\phi$	Def.,4
6	$\Box\neg\phi$	R5),5
7	$\Box\phi$	R),2
8	~~Show~~: $\Box\Diamond\phi$	[Nec,11]
9	$\neg\phi$	R) (**K**),6
10	ϕ	R) (**K**),7
11	$\Diamond\phi$	$\neg$E),9,10
12	$\Diamond\phi$	$\Box E^{\Diamond}$),3

Remark 6. Because in this paper we consider only some limited group of logics for which the Fitch's approach works quite well, one can be interested in its real scope. Hawthorn [14] claimed that modification of R), works well only for those modal logics which are axiomatizable with the help of formulae of the kind $\sigma\phi \rightarrow \Box\delta\phi$, where σ and δ are finite strings of $\Box$ and $\Diamond$. It is not true however; [17] contains the definition

of Γ^*, for many temporal logics, axiomatized by formulae of different shape. Also some considerations on additional devices allows to extend this kind of ND-formalization. By the way of example the adequate system for **G** may be obtained taking **DN-K4** and an additional rule for adding modal assumption:

c) if the last show-formula is $\Box\phi$, then we may add $\Box\phi$ as a *modal assumption* of that derivation.

Closing by [Nec], with this additional device is justified in **G** because this logic satisfies the condition:

if $\Gamma^*, \Box\phi \vdash \phi$, then $\Gamma \vdash \Box\phi$.

Remark 7. The definition of Γ^* may be simplified in some cases because it is redundant. **B** and **S5** can serve as examples. Let us define Γ^*(**B**) as $\{\Diamond\phi{:}\phi \quad \Gamma\}$ and Γ^*(**S5**) as $\{\Diamond\phi{:}\Diamond\phi \quad \Gamma\}$. Admissibility is obvious, so it is enough to prove sufficiency of these definitions. Provability of MP), RG) and T) is intact, B) is still derivable trivially with this Γ^* in both logics. Notice that in both logics we have also secondary inference rule:

RB) $\Diamond\Box\phi \vdash \phi$, the proof of which is obvious, and $\Diamond$I) $\phi \vdash \Diamond\phi$. Below we present a proof of K) in **DN-B**, with weakened R):

1	~~Show~~:$\Box(p \to q) \to (\Box p \to \Box q)$	[Cond,3]
2	$\Box(p \to q)$	assumption
3	~~Show~~:$\Box p \to \Box q$	[Cond,6]
4	$\Box p$	assumption
5	$\Box(p \to q)$	R),2
6	~~Show~~:$\Box q$	[Nec,11]
7	$\Diamond\Box(p \to q)$	R) (**B**),5
8	$\Diamond\Box p$	R) (**B**),4
9	$p \to q$	RB),7
10	p	RB),8
11	q	$\to$E),9,10

Analogously we can prove K) in **DN-S5**, the only difference is that in line 4 and 5, we should first apply $\Diamond$I), before it is possible to put them by R) to the strict derivation. In **S5** one must also prove 4):

1	~~Show~~: $\Box p \rightarrow \Box\Box p$	[Cond,4]
2	$\Box p$	assumption
3	$\Diamond\Box p$	$\Diamond$I),2
4	~~Show~~: $\Box\Box p$	[Nec,6]
5	$\Diamond\Box p$	R) (**S5**),3
6	~~Show~~: $\Box p$	[Red,12,13]
7	$\neg\Box p$	assumption
8	$\Diamond\neg p$	Def.,7
9	~~Show~~: $\Box\neg\Box p$	[Nec,11]
10	$\Diamond\neg p$	R) (**S5**),8
11	$\neg\Box p$	Def.,10
12	$\neg\Diamond\Box p$	Def.,9
13	$\Diamond\Box p$	R),5

One could easily notice that in this way it is possible also to get DN for **KB** and **KDB**, simply dropping $\Box$E), or replacing it by D) $\Box\phi \vdash \Diamond\phi$. It is not possible to formalize **KB4** in this way. **KD4E** allows for a weaker definition of Γ^* too because the above proof of 4) is also a proof in this logic (line 3 is justified by $\Diamond I^{\Box}$), which is dual version of $\Box E^{\Diamond}$)), so it is sufficient to define Γ^*(**KD4E**) as a sum of Γ^* for **K** (instead of **K4**) and for **S5** in the latter version.

2.3. Modalization of rules

Segerberg and Bull [6] proposed another method for dealing with modal logics in ND-systems. The starting point is the observation that any rule valid in **CPC** should be still valid in any modal context. The notion of a context is then explained in the following way: if $\Gamma \vdash \phi$ is valid in **CPC**, then the addition of n $\Box$ to all elements of Γ and ϕ makes the rule still valid. Essentially it is the application of the condition $\Box$) which is also fundamental for Fitch approach, where it is a theoretical basis for [Nec] together with R). Here $\Box$) is a justification of a modalization of any inference rule. For example, if in any F-system presence of ϕ and $\phi \rightarrow \psi$ in the proof allows to add ψ by $\rightarrow$E), then by condition $\Box$), we can add to proof $\Box^n\psi$, if we have already $\Box^n\phi$ and $\Box^n(\phi \rightarrow \psi)$. In a similar way we can modalize all other inference rules, e.g. for   :

$\Box^n$(I) $\Box^n\phi$, $\Box^n\psi \vdash \Box^n(\phi \quad \psi)$

$\Box^n$(E) $\Box^n(\phi \quad \psi) \vdash \Box^n\phi$ (ψ)

Segerberg did more, because he also modalized all proof rules. Such solution cannot be justified by condition □), because it is sufficient only for inference rules. Nevertheless it is in accordance with the starting motivation. Modalization of $\rightarrow$I) and $\neg$E) is based on the following principles:

if $\Gamma^{*n}, \phi \vdash \psi$, then $\Gamma \vdash \Box^n(\phi \rightarrow \psi)$

if $\Gamma^{*n}, \neg\phi \vdash$, then $\Gamma \vdash \Box^n\phi$

(where Γ^{*n} means $\{\phi : \Box^n\phi \quad \Gamma\}$, both are derivable by **CPC** and □))

In Segerberg's system there are no introduction and elimination rules for □. In case n=0, rules are simply **CPC**-rules, and the whole system is adequate for **K**. Segerberg suggests that such basic system for key-logic is enough; all extensions should be obtained by axiomatic additions. Despite this we can consider whether in such a system some extensions can be obtained in a different way. The other problem is what representation fits to such system. Modalized inference rules may be applied in any format; of course, if it is a S-system, then their content-items are not sequents, but sequent rules, for example for we have:

$\Box^n($ $I^S)$ if $\Gamma \vdash \Box^n\phi$ and $\Delta \vdash \Box^n\psi$, then $\Gamma, \Delta \vdash \Box^n(\phi \quad \psi)$

$\Box^n($ $E^S)$ if $\Gamma \vdash \Box^n(\phi \quad \psi)$, then $\Gamma \vdash \Box^n\phi\ (\psi)$

The problem arises with modalized proof rules. Segerberg suggested F-T-system but it is not clear how, in practice, one should mark in such a proof the transition from Γ^{*n} to Γ. If we use Jaśkowski format it is evident that, in fact, we are applying R) in its modalized version, hence the system is quite similar to Fitch's approach, with the only difference that there is no special rule [Nec] because strict derivations may be entered by [Cond] and [Red] with the addition of □ n>0 times to show-formula.

It seems that Segerberg's system may be simply modified to obtain universal formalization, independent of the basic format, and allowing extensions for logics other than **K**. The point is the redundancy of the system. First of all we can always keep n=1 in the indices of □ in the definition of rules, moreover we may resign from the modalization of many rules. One of the possible solution is to modalize proof rules only, which is an obvious implication of the preceding paragraph. If we have R) in the variant for **K**, such limitation is sufficient; we may even use solely modalized [Red]. In practice it is realised in such a way that to any proof in a system **DN-K**, from preceding section we add as an assumption $\neg\phi$, under each show-line $\Box\phi$ and change justification from [Nec] to [Red$^{\Box}$]. So modified Segerberg's system is not very original; it is in fact a variant of Fitch's system.

A better solution, still in accordance with original motivation is to limit modalization only to inference rules. It is based on natural interpretation of condition $\Box$) and ND-system thus obtained is not a variant of Fitch's system any more because it forces to use different proof strategies. To get an adequate formalization of **K** without any modification of proof rules, one must allow also a modalization of inference rules with empty set of premises, which is simply an application of GR). Although formalization of this kind is universal, practically it is simpler to use it with S-format because of the last proviso; in Jaśkowski format it is not immediately evident if a formula is really not dependent on any assumptions.

A separate problem is the possibility of extensions of this formalization for other logics. A proposal of this kind may be found in Hawthorn [14] but we omit wider presentation because of limited space and because it seems to be not very useful from the practical point of view. Many normal logics may be formalised with relatively smaller effort. Instead we shall pay attention to modalization of theses and consider some variants of GR):

GR$^{\Box}$) You may put $\Box$ before ϕ, if the set of assumptions for ϕ is empty or ϕ is a $\Box$-formula.

GR$^{\Diamond}$) You may put $\Box$ before ϕ, if the set of assumptions for ϕ is empty or ϕ is a $\Diamond$-formula.

GR$^{\Box\Diamond}$) You may put $\Box$ before ϕ, if the set of assumptions for ϕ is empty or ϕ is a M-formula.

It is evident that GR$^{\Box}$) gives **K4**, GR$^{\Diamond}$) **KE**, and GR$^{\Box\Diamond}$) **K4E**. Addition of rules $\Box$E$\Diamond$) or $\Box$E) allows to obtain a formalization for **D, T, KD4, S4, KDE, KD4E, S5**. Obviously we resign in this way from the idea that there are no specific rules for modal functors but it seems that adding axioms to **K** is not better.

Remark 8. The system where only inference rules are modalized is still redundant. It is sufficient to keep as primary rule $\Box^n(\rightarrow E)$, and GR), all other modalized variants of inference rules of the sort $\{\psi 1,\ldots\psi n\} \vdash \phi$ may be easily derivable as follows:

{1}	1	$\Box^n \psi 1$	
:	:	:	
{n}	n	$\Box^n \psi n$	
	n+1	$\psi 1 \rightarrow (\psi 2 \rightarrow \ldots(\psi n \rightarrow \phi)\ldots)$	by applying n-times $\rightarrow$I) to the rule in question
	n+2	$\Box^n(\psi 1 \rightarrow (\psi 2 \rightarrow \ldots(\psi n \rightarrow \phi)\ldots))$	by GR)
:	:	:	
{1,...,n}	n+2+n	$\Box^n \phi$	1-n,n+2, n times $\Box^n(\rightarrow$E)

Remark 9. **KD4E** may be simply formalized on the base of a system with modalized inference rules; it is enough to add GR$^\Diamond$) and $\Box$E$^\Diamond$).

3. The possibilistic approach

So far we have used $\Diamond$ as a definitional shortcut, favouring $\Box$, which was in accordance with the usual practice of many authors. However the problem of formalization of $\Diamond$ is sufficiently interesting in itself to be described separately. In the original system of Fitch $\Diamond$ was in fact treated as an independent functor and characterised on the level of content by the pair of rules:

$\Diamond$I) $\phi \vdash \Diamond\phi$, only for reflexive logics

$\Diamond$E) if $\Gamma^*, \psi \vdash \phi$, then $\Gamma, \Diamond\psi \vdash \Diamond\phi$.

In **K&M** two rules correspond to $\Diamond$E); one for closing a derivation and one for entering assumptions:

[Poss] Let $\Diamond\phi$ be the show-formula of k-degree derivation and the first usable-formula be a *modal* assumption, then we can close this derivation, provided ϕ has appeared as a usable-formula in this derivation.

d) if $\Diamond\psi$ is a usable-formula of k-degree derivation and $\Diamond\phi$ is a show-formula of k+1-degree derivation, then we may add ψ as a *modal assumption* of the k+1-degree derivation.

Also R) should be modified, we introduce the version which works for the system in which both [Nec] and [Poss] are primitive. Because of stylistic reasons it will be defined dually to our official R) from paragraph 2.2. :

Let Γ be the set of usable-formulae of k-degree derivation and ϕ Γ^*, we may put ϕ into subderivation of k+1-degree by R) if at least one of the conditions is satisfied:
a) the show-formula is $\Box\psi$ and the first usable-formula of this derivation is not an indirect assumption.
b) the first usable-formula is a modal assumption.
If neither a) nor b) is satisfied, then we may put into this derivation any ϕ Γ.

Schematically it looks like this:

	:		
i-j	Γ		
	:		
k	$\Diamond\psi$		
l	~~Show~~:$\Diamond\phi$		[Poss,o]
l+1		ψ	assumption
m-n		Γ^*	R)i-j
		:	
o		ϕ	i+1, m-n

One should notice that although an introduction of modal assumption is an optional element (we may have some $\Diamond$-formula as show-formula but to try to close this derivation by different rule), its presence is a necessary condition to close the derivation by [Poss].

The system of Fitch contains also four rules of elimination and introduction for negated modal formulae of the sort:

$\neg\Box$E)	$\neg\Box\phi \vdash \Diamond\neg\phi$
$\neg\Box$I)	$\Diamond\neg\phi \vdash \neg\Box\phi$
$\neg\Diamond$E)	$\neg\Diamond\phi \vdash \Box\neg\phi$
$\neg\Diamond$I)	$\Box\neg\phi \vdash \neg\Diamond\phi$

The presence of these rules, let us call them *definitional rules*, raises the question: are they really necessary? If not, then how to formulate ND-system with primitive rules for $\Box$ and $\Diamond$ which are really independent and sufficient to derive all definitional rules? Such a solution, if possible, will be in accordance with the normal practice of treating logical constants in ND and other systems like sequent calculus. Although and are interdefinable, usually they are characterised by the pairs of independent rules (in particular is not present in the rules for , and vice versa); De Morgan’s rules are then introduced as derivable. The wish to have such a characterization for modal connectives is fully justified. The problem is of a general character (see e.g. [28] or [18]). and perhaps, for the first time noted by Routley [24], where incompleteness of modal rules in the sequent system of Ohnishi and Matsumoto [22] was noted. We will take a closer look at this question in the next paragraph.

For the most part of this section we will use **K&M** system for **CPC** as basic, the addition of some modal rules will be explicitly marked, e.g. **CPC**+[Nec] means that it is a system for **CPC** with [Nec].

3.1. Interdefinability problem

The system of Fitch is redundant indeed, but in a different, not very satisfying way. We cannot get rid of all definitional rules, but we can use only some of them because they are interderivable in the following way:

Lemma 3. a) ¬◇E) *is derivable in* **CPC**+[Poss]+¬□E)
b) ¬◇I) *is derivable in* **CPC**+[Poss]+¬□I)
c) ¬□E) *is derivable in* **CPC**+[Nec]+¬◇E)
d) ¬□I) *is derivable in* **CPC**+[Nec]+¬◇I)

Proof. We will put schemes of derivation for a) and d); b) and c) analogously

1	¬◇ϕ	
2	~~Show~~:□¬ϕ	[Red,5,8]
3	¬□¬ϕ	assum.
4	◇¬¬ϕ	¬□E),3
5	~~Show~~:◇ϕ	[Poss,7]
6	¬¬ϕ	assum.
7	ϕ	DN),6
8	¬◇ϕ	R),1

1	◇¬ϕ	
2	~~Show~~:¬□ϕ	[Red,4,8]
3	□ϕ	assum.
4	◇¬ϕ	R),1
5	~~Show~~:□¬¬ϕ	[Nec,7]
6	ϕ	R) (**C**),3
7	¬¬ϕ	DN),6
8	¬◇¬ϕ	¬◇I),5

Simple consequence of this lemma is the observation that in order to have a full system with □ and ◇ as primitive it is enough to use **K&M** with [Nec] and [Poss] (and possibly with □E) and ◇I)) and any of the four pairs: ¬□E) and ¬□I), or ¬◇E) and ¬◇I), or ¬□I) and ¬◇E), or ¬□E) and ¬◇I).

On the other hand, the primitive rules for □ and ◇ are not independent: □E) and ◇I) are interderivable in the system with ¬◇E) (for a proof of ◇I)) and in the system with ¬□I) (for a proof of □E)). What is more also [Nec] and [Poss] are in fact mutually dependent, although not, in general, interderivable. We can prove the following lemma:

Lemma 4. a) [Poss] *is admissible in* **CPC**+[Nec]+¬◇E)+¬◇I);
b) [Nec]^{REG} *is admissible in* **CPC**+[Poss]+¬□E)+¬□I).

(where [Nec]^{REG} means [Nec] with proviso for regular logics)

The proof is involved in the following schemes of elimination:

	:	
i-j	Γ	
k	◇ψ	
	:	
l	~~Show~~:◇φ	[Poss,o]
l+1	ψ	assum.
m-n	Γ*	R),i-j
	:	
o	φ	i+1, m-n

	:	
i-j	Γ	
k	◇ψ	
	:	
l	~~Show~~:◇φ	[Red,l+3,r+2]
l+1	¬◇φ	assumption
l+2	□¬φ	¬◇E),l+1
l+3	◇ψ	R),k
l+4	~~Show~~:□¬ψ	[Nec, n+2]
m-n	Γ*	R),i-j
n+1	¬φ	R) (**C**),l+2
n+2	~~Show~~:¬ψ	[Red,r,r+1]
n+3	ψ	assumption
o-p	Γ*	R),m-n
	:	
r	φ	by assumed proof; n+3,o-p
r+1	¬φ	R),n+1
r+2	¬◇ψ	¬◇I), l+4

	:	
i-j	Γ	
k	$\Box\psi$	
	:	
l	~~Show~~:$\Box\phi$	[Nec,o]
m-n	Γ^*	R),i-j
	:	
	:	
o	ϕ	m-n

i-j	Γ	
k	$\Box\psi$	
	:	
l	~~Show~~:$\Box\phi$	[Red,s+2,s+3]
l+1	$\neg\Box\phi$	assumption.
l+2	$\Diamond\neg\phi$	$\neg\Box$E),l+1
m-n	Γ	R),i-j
o	~~Show~~:$\Diamond\neg\psi$	[Poss,s+1]
o+1	$\neg\phi$	assumption
p-r	Γ^*	R),m-n
	:	
s	ϕ	by assumed proof; p-r
s+1	$\neg\psi$	$\neg$E),o+1,s
s+2	$\neg\Box\psi$	$\neg\Box$I),o
s+3	$\Box\psi$	R),k

It is evident that [Nec] and [Poss] are equivalent in regular logics; [Poss] is too weak to capture in ND the effect of GR) but we can use a variant of [Poss] which is sufficiently strong. Wisdome [29] proposed a rule $[\ \ ]^W$ and Fitting [11] $[\ \ ]^F$:

$[\ \ ]^W$ if $\Gamma^*, \psi \vdash$, then $\Gamma, \Diamond\psi \vdash \phi$

$[\ \ ]^F$ if $\Gamma^*, \psi \vdash$, then $\Gamma, \Diamond\psi \vdash$

Both variants are equivalent; $[\ \ ]^W$ follows from $[\ \ ]^F$ in **CPC**, and the latter is a particular form of $[\ \ ]^W$, hence further we will simply use [ ] for any of them. The equivalence of [Nec] and [ ] is stated in the next lemma:

Lemma 5. a) [ ] *is admissible in* **CPC**+[Nec]+$\neg\Diamond$I);
b) [Nec] *is admissible in* **CPC**+[ ]+$\neg\Box$E).

Proof by similar schemas as in the previous lemma.

Remark 10. There is no need to modify [Poss] in regular logics because sequent rule $\Diamond$E) is satisfied in each regular logic, even with empty Γ.

On the other hand in case of normal logic **D** there is no need to add modal assumption in the strict derivation initiated by [Poss], because in **D** we have admissible rule:

if $\Gamma^* \vdash \psi$, then $\Gamma \vdash \Diamond\psi$.

Hence, it is possible in all extensions of **D** to simplify [Poss], making an introduction of modal assumption really an optional element, similarly as in [Cond] and [Red]. Moreover it lets to obtain adequate formalization for **D**, without any special rule of inference such as D) $\Box\phi \vdash \Diamond\phi$, which is now simply derivable. If we allow such a simplification in the definition of [Poss] on the ground of regular logics we must add similar proviso as in the case of the definition of [Nec] in them, namely that in outer derivation at least one usable-formula should be a $\Box$-formula.

In effect, instead of the expected system with independent rules we have a different opportunity: a system with $\Diamond$ as basic functor based on [ ] (or [Poss], if we restrict our interests to regular logics). We will go back to this question in the next paragraph. Anyway the choice of ND based on [Nec] or on [ ] still forces us to add some of the definitional rules as primitive. Possible choices are taken together in the next lemma:

Lemma 6. *Adequate formalization of* **C** (**K**) *on the basis of* **CPC** *may be obtained by the addition of* a) [Nec]+¬$\Diamond$I)+¬$\Diamond$E) *or* b) [ ]+¬$\Box$E)+¬$\Diamond$E).

Proof. a) by lemma 3, c), d)
for b) the proof of ¬$\Box$I); for ¬$\Diamond$I) the proof is analogous.

1	$\Diamond\neg\phi$	assumption
2	~~Show~~: $\neg\Box\phi$	[Red,5]
3	$\Box\phi$	assumption
4	$\Diamond\neg\phi$	R),1
5	~~Show~~:	[,6,7]
6	$\neg\phi$	assumption
7	ϕ	R) (**C**),3

The drawback of the latter system is that [ ] is not a good representative of modal rule, which is especially evident in **K&M**, where it looks like a kind of indirect proof, and only the rule of introduction of modal assumption has some flavour of modal rule but it is independent of [   ], at least on the level of representation.

Now we are ready to answer the question, if it is possible to get a system, where all definitional rules are derivable. [11] contains formalizations of this sort, but this solution is based on the modification of R) in a way that the interdefinability of $\Box$ and $\Diamond$ is already contained in the definition of the Γ^*, e.g. for **C** (and other nontransitive logics) it looks $\{\phi:\Box\phi \ \ \Gamma\} \ \{\neg\phi:\neg \ \Diamond\phi \ \ \Gamma\}$ and analogously for other logics. It is obvious that all definitional rules are derivable with the help of such a definition of Γ^*, both with the use of [Nec], and [Poss]. The disadvantage is in a growing complexity of definition for Γ^*, what is more, we must also generalise [Nec], so that it is possible to close a derivation starting with "Show:$\neg\Diamond\phi$" provided we deduce usable-formula $\neg\phi$; in [   ] we can add as a modal assumption $\neg\phi$ if we have already $\neg\Box\phi$ in the proof. These rather radical changes are partly obscured in [11] by the use of generalized terminology. It seems that the problem is solved in a satisfying way only in the deductive systems of a very generalised sort, where radical enrichment of the basic structural tools makes possible greater flexiblity e.g. in *Display Logic* (see [4], [28]) or in *multisequential sequent calculus* (see [18]). But one should notice that, at least with respect to some logics, like **S4**, or **S5**, we can get satisfying result by very modest changes, without loosing standard ND-format and changing [   ] by the variant of [Poss] much better suited as a representative of basic rule for $\Diamond$. System of this sort for **S5** is present in [15] but it is also possible to make some changes in order to have a system for **S4.** In a system for **S5**, we admit that R) in strict derivations is limited to M-formulae, [Nec] is in standard form and [Poss] is based on the following rule from the content:

if $\Gamma^*, \psi \vdash \phi$, then $\Gamma , \Diamond\psi \vdash \phi$, where ϕ is any M-formula.

Variant for **S4** is similar but R) in case of [Nec] and [Poss] is restricted to $\Box$-formulae and ϕ in the schema of [Poss] must be any $\Diamond$-formula. Proofs of definitional rules are the following:

1	$\Box\neg\phi$	
2	~~Show~~: $\neg\Diamond\phi$	[Red,4,5]
3	$\Diamond\phi$	assum.
4	$\Box\neg\phi$	R),1
5	~~Show~~: $\neg\Box\neg\phi$	[Poss,10]
6	ϕ	assum.
7	$\Box\neg\phi$	R) (**S4**),4
8	$\neg\phi$	$\Box$E),7
9		$\neg$E),6,8
10	$\neg\Box\neg\phi$	$\neg$E),9

1	$\neg\Box\phi$	
2	~~Show~~: $\Diamond\neg\phi$	[Red,4,5]
3	$\neg\Diamond\neg\phi$	assum.
4	$\neg\Box\phi$	R),1
5	~~Show~~: $\Box\phi$	[Nec,7]
6	$\neg\Diamond\neg\phi$	R) (**S4**),3
7	~~Show~~: ϕ	[Red,9,10]
8	$\neg\phi$	assum.
9	$\Diamond\neg\phi$	$\Diamond$I),8
10	$\neg\Diamond\neg\phi$	R),6

Proof of $\neg\Box$I) is analogous to the proof of $\neg\Diamond$I) and a proof of $\neg\Diamond$E) is in fact contained in the above proof of $\neg\Box$E) (lines 3 and 5-10).

So in both cases of sufficiently strong logics we can get systems with four basic rules for $\Box$ and $\Diamond$, (although not independent) and without definitional rules; [Poss] is interpreted as a rule of elimination for $\Diamond$ (because of the way of introducing modal assumptions).

3.2. Modal assumptions

In the previous paragraph we have noticed that adequate formalization of modal logics can be based on $\Diamond$ as a primitive functor. This possibility is fully realized by Fitting [11], who presented two ND-systems for modal logics, one called *A-system*, is based on [Nec] and the second, *I-system*, is based on [ ]. This distinction is quite important to Fitting, who remarked that in proof construction A-system is more connected with axiomatic formalizations with GR), I-system is rather close to tableau systems. Semantic reasons are even more important; strict derivations closed by [Nec] are interpreted in a different way than those closed by [ ]. In the former case the separated derivation is a counterpart of an arbitrary chosen world (hence the name A-system per analogiam to traditional general-categorical A-statements). In the latter it is a counterpart of some specific world in a Kripke model (hence the name I-system as particular-categorical I-statement). This informal interpretation is for Fitting a base for the construction of the respective soundness proofs for both systems.

So far we have run a different course, of mixing both systems but full characterisation apart, we can ask a question: what advantages our possibilistic approach, "possibly" offers? Previous presentation presupposed Jaśkowski format with the apparatus of strict derivation and restricted reiteration. But the presence of modal assumptions can make at least one of these components unnecessary, because all the necessary information which was transported into strict derivation by R) may be incorporated into the modal assumption, which is possible due to the properties of logics. Let us display in the next lemma some new rules of ◇I) derivable in some regular logics (and their extensions):

Lemma 7. *The following rules are derivable in the respective logics:*

a) $\Diamond\phi$, $\Box\psi \vdash \Diamond(\phi \quad \psi)$, *for any logic*
b) $\Diamond\phi$, $\Box\psi \vdash \Diamond(\phi \quad \Box\psi)$, *for any transitive logic*
c) $\Diamond\phi$, $\psi \vdash \Diamond(\phi \quad \Diamond\psi)$, *for any symmetric logic*
d) $\Diamond\phi$, $\Diamond\psi \vdash \Diamond(\phi \quad \Diamond\psi)$, *for symmetric and reflexive logic*

Proof is an easy exercise.

The similarity of these rules to the respective definitions of Γ* is straightforward. It seems that whenever we have a definition of Γ*, we can define suitable ◇I) for exactly the same logics, but now formulated with [ ] instead of [Nec]. It is evident that we can get rid of restricted R) in a system where we use above rules as primitive. The only way of entering strict derivation is [ ] and R) is simply forbidden here. All formulae that we need to close such a proof, and that were so far transported by R), are now linked into conjunction with ◇, which is next turned into modal assumption. But we can go even further because such a solution is not longer dependent on Fitch's approach. We do not have special reasons to separate parts of the proof as subderivations, strict or nonstrict or whatever, hence we can combine this approach with any format. Let us illustrate how to get such a system for **C** in Suppes representation. As a counterpart of [Poss] we will introduce the following rule for ◇E):

π	i	$\Diamond\phi$	
	.		
	.		
{k}	k	ϕ	modal assumption
	.		
	.		
{k}	n	ψ	
π	n+1	$\Diamond\psi$	i, k, n, ◇E), where π is the set of numbers of assumptions for $\Diamond\phi$

It is justified by the following rule admissible in all considered logics:

if $\Gamma \vdash \Diamond\varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \Diamond\psi$.

We also need a sequent version of the first of $\Diamond$I) rules, displayed above in lemma 7 and some sequent versions of definitional rules. In order to get the extensions of **C** it is enough to add more suitable sequent versions of the above rules to the system and to strengthen our version of $\Diamond$E) , replacing ψ by , especially if we want to formalize normal logics (see lemma 6, b)). Careful reader should not have any problems with detailed exposition, so we keep it as granted. The overall moral is that $\Diamond$ seems to fit better to ND-systems as a primitive functor because it is format insensitive.

University of Łódź, indrzej@krysia.uni.lodz.pl

REFERENCES

[1] Anderson, A.R., Belnap N.D., *Entailment: the Logic of Relevance and Necessity*, vol. 1, Princeton University Press, Princeton 1975.

[2] Avron A., Honsell F., Miculan M., Paravano C., "Encoding Modal Logics in Logical Frameworks", *Studia Logica*, 60, 1998, 161-208.

[3] Basin D., Matthews S., Vigano L., "Natural Deduction for Non-Classical Logics", *Studia Logica*, 60, 1998, 119-160.

[4] Belnap N.D., "Display Logic", *Journal of Philosophical Logic*, 11, 1982, 375-417.

[5] Borkowski L., Słupecki J., "A logical system based on rules and its application in teaching mathematical logic", *Studia Logica*, 7, 1958, 71-113.

[6] Bull R., Segerberg K., "Basic Modal Logic", in D. Gabbay, F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. II, 1-88, Reidel, Dordrecht, 1984.

[7] Corcoran J., Weaver G., "Logical consequence in modal logic: Natural Deduction in S5", *Notre Dame Journal of Formal Logic*, 10, 1969, 370-384.

[8] Curry H. B., *A theory of formal deducibility*, University of Notre Dame Press, Notre Dame, 1950.

[9] Fitch F. B., *Symbolic Logic: An Introduction*, Ronald Press, New York, 1952.

[10] Fitch F. B., "Natural deduction rules for obligation", *American Philosophical Quaterly*, 3, 1966, 27-38.

[11] Fitting M., *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Dordrecht, 1983.
[12] Gentzen G., “Untersuchungen über das logische Schliessen”, *Mathematische Zeitschrift*, 39, 1934, 176-210, 405-431.
[13] Gentzen G., “Die Widerspruchsfreiheit der reinen Zahlentheorie”, *Mathematische Annalen*, 112, 1936, 493-565.
[14] Hawthorn J., “Natural Deduction in Normal Modal Logic”, *Notre Dame Journal of Formal Logic*, 31, 1990, 263-273.
[15] Hazen A., “Actuality and Quantification”, *Notre Dame Journal of Formal Logic*, 31, 1990, 498-508.
[16] Indrzejczak A., “Natural Deduction in Quantifier Modal Logic”, *Bulletin of the Section of Logic*, vol. 23/1, 1994, 30-40.
[17] Indrzejczak A., “Natural Deduction System for Tense Logics”, *Bulletin of the Section of Logic*, vol. 23/4, 1994, 173-179.
[18] Indrzejczak A., “Generalised Sequent Calculus for Propositional Modal Logics”, *Logica Trianguli*, 1, 1997, 15-32.
[19] Indrzejczak A., “Jaśkowski and Gentzen approaches to Natural Deduction and related systems”, in K. Kijania-Placek, J. Woleński, (eds.), *The Lvov-Warsaw School and Contemporary Philosophy*, Kluwer Academic Publishers, 1998, 253-264.
[20] Jaśkowski S., “On the rules of suppositions in formal logic”, *Studia Logica*, 1, 1934, 5-32.
[21] Kalish D., Montague R., “Remarks on Descriptions and Natural Deduction”, *Archiv. für Mathematische Logik und Grundlagen Forschung*, 3, 1957, 50-64, 65-73.
[22] Ohnishi M., Matsumoto K., “Gentzen Method in Modal Calculi”, *Osaka Mathematical Journal*, 9, 1957, 113-130.
[23] Prawitz D., *Natural Deduction*, Almqvist & Wiksell, Stockholm 1965.
[24] Routley R., review of [22], *Journal of Symbolic Logic*, 40, 1975, 466-467.
[25] Siemens D. F., “Fitch-style rules for many modal logics”, *Notre Dame Journal of Formal Logic*, 18, 1977, 631-636.
[26] Suppes P., *Introduction to Logic*, Van Nostrand, Princeton 1957.
[27] Thomason R. H., “A Fitch-style formulation of Conditional Logic”, *Logique et Analyse*, 13, 1970, 397-412.
[28] Wansing H., “Sequent Calculi for Normal Modal Propositional Logics”, *Journal of Logic and Computation*, 4, 1994, 125-142.
[29] Wisdome W. A., “Possibility elimination in natural deduction”, *Notre Dame Journal of Formal Logic*, 5, 1964, 295-298.