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BASIC LOGIC FOR ONTIC AND DEONTIC MODALITIES

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Abstract

The difficulty to interpret the iteration of modalities, already ontic and still more deontic, incites to pay attention to the system B of *basic modal logic* that John L. Pollock proposed in 1967. The Pollock's system brought all the theses which, in the classical ontic modal systems, from Sl to S5, contain no iteration of the modal functors. With this *basic ontic system* we characterize a *basic deontic system*, and a *basic ontico-deontic system*, the former including all the theses of the first two. Each of the three systems is based axiomatically and assorted with a semantics for which the soundness and the completeness hold.

In [6] John L. Pollock proposed a *basic modal logic*, B, whose main characteristics was that it did not admit any iterated modality, i.e. no modal operator would appear within the scope of another modality. The system B[1] may be based on two axioms:

A1 $\quad$ $Lp \quad p$

A2 $\quad$ $L(p \quad q) \quad (Lp \quad Lq)$

and the four inference rules:

R1 *Rule of substitution*, proviso that the substitution of a WFF (well formed formula) does not generate any iteration of modality

R2 *Rule of replacement*, with the classical definitions of PC (propositional calculus) and the definition

$$Mp =_{def} \quad L \quad p$$

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[1] To simplify, we slightly change Pollock's presentation of B. Obviously not changing its content.

R3 *Rule of detachment*

R4 *Rule of necessitation*

$\models_{PC} \alpha \qquad \vdash_{B} L\alpha$

(i.e. if α is valid in PC, then $L\alpha$ is a thesis of B).

For this system Pollock defined a validity which permitted to establish the soundness of the system as well as its completeness. He rightly pointed out that the iteration of the modalities bore great difficulties of interpretation, and that, for most practical cases, a modal logic without iterations was sufficient. Pollock succeeded in establishing that Lewis's classical modal systems, from S1 to S5, as well as the system M of von Wright (sometimes called system T, following Sobociński) which admit the iteration of modalities, not only all contain B, but they do not have any other thesis written without iteration than those of B. He qualified these systems as *regular*; and thus a regular system is this in which the set of the theses without any iteration of modalities coincides with B. Finally, he also proved that no proper extension of S5 was regular.

In this way Pollock showed that the system B is common for most usual modal systems. The interest of B affects also S5 for at least two reasons:

1) for every WFF of S5 it can be found in S5 an equivalent formula of the first modal degree (i.e. without any iteration of modalities) of B. Due to this, every thesis of S5 may be related to a thesis being its translation in B;[2]
   the fact that a regularity of proper extension of S5 is impossible explains why the extensions of this system evoke little interest; therefore S5 naturally marks a sort of stopping-point in the hierarchy of the modal systems[3].

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[2] On the other hand, different theses of S5 may have the same corresponding thesis of the first modal degree. Thus, e.g.

$\vdash_{S5} LLp \quad p$

$\vdash_{S5} LLLp \quad p$

$\vdash_{S5} LLLLp \quad p$, etc.

have all the same corresponding thesis

$\vdash_{S5} Lp \quad p$.

[3] Cf. Hughes & Cresswell [5], p. 346. In this diagram, B should occur after SO.5, i.e. after all other systems. The authors explain (p. 256, footnote 245) why they have not discussed such a system in their book.

Here I will show that the advantages of the modal system B, such as acknowledged by Pollock, go beyond the simple ontic modalities (*necessary, impossible, possible, contingent*) and may be found in a modal system admitting also the deontic modalities.

First, let us observe that the modality iteration raises more difficulties of interpretation in the deontic realm than in the ontic one. In the ontic field, indeed, a semantics of possible worlds allows certain interpretation of the iteration; it is only in its application to ordinary reasoning that the interpretation of the iteration seems to be difficult. In the deontic field, on the contrary, the only interpretation that was proposed of the iteration concerns the hierarchical norms. But an expressions like:

It is obligatory that it is obligatory that p

is apparently meaningful only if the author of the first obligation is not the same as (but hierarchically superior to) the subject from whom the second obligation comes. Here speaking of *iteration* amounts to forgetting that the first obligation neither proceeds from the same authority nor applies to the same subject as the second obligation. A logic of the hierarchical norms is plainly possible under the condition that the deontic functor does not have a unique argument, designating the performance to accomplish (like the classical ontic modal functor), but three arguments, that is, beside the preceding one, two names designating an individual from whom the norm is emanating and one to whom it applies. As long as we lack the vocabulary necessary to analyse hierarchical situations and we are limited to use a modal functor with a unique propositional argument, it is important, much more for deontic modalities than for ontic ones, to adopt rules of good formation of expressions forbidding the iteration of modal functors.

Let us start from the very system B, and build a semantics corresponding to it, i.e. according to which all theses of B are valid and all valid expressions are theses of B. We shall consider a first world W_0, accessible to itself, and an indeterminate number of worlds W_1, W_2, etc. accessible to W_0. We shall translate semantically "Lα is true in W_0" as follows:

α *is true in every world accessible to* W_0

and "Lα is false in W_0" as:

There is at least one world accessible to W_0 *in which* α *is false*.

The classical definition:

$Mp =_{df}$ L p

will then permit to derive forms for "Mα is false in W_0":

α *is false in every world accessible to* W_0

and for "Mα is true in W_0":

There is at least one world accessible to W_0 *in which* α *is true.*

It can be easily shown that both axioms of B are valid, since they cannot be falsified. The validity of A1 can be established by showing that if "Lp p" were false in W_0, then "Lp" would be true and "p" false in it; but the truth of "Lp" in turn involving the truth of "p" (since W_0 is accessible to itself), "p" would be in W_0 true and false, which is contradictory. It is also possible to check that if A2 were false in W_0, there would be some world accessible to W_0 in which "q" would be both true and false. It can be easily established that the four rules preserve validity. Therefore all theses of B are valid.

To establish that all valid expressions are theses of B, i.e. to prove the completeness of B, is evidently more difficult[4]. For this purpose I shall represent every world considered in the reasoning by a tableau of two-columns the left for the expressions admitted as true, the right for ones admitted as false, beginning with an expression to validate, since the apagogic reasoning which constitutes the process of validation begins with supposing that this expression is false in the initial world W_0.

Let us first eliminate from the expression to validate every modal operator other than L. Thus each tableau will first receive one expression in its right and n ($n \geq 0$) expressions $\alpha_1, \alpha_2, \ldots, \alpha_n$ in its left. We shall call *initial formula* of the tableau in question the formula:

$$(\alpha_1 \quad \alpha_2 \ldots \alpha_n) \quad \beta$$

which is reduced to β if n=0, which is evidently the case for W_0.

We first shall show that:

[4] Although the completeness was already proved by Pollock, I propose here another demonstration, since I shall need precisely such validity as I have characterized in order to build the other forms of validity and the other demonstrations of completeness which the introduction of deontic functors will lead us to consider.

Every initial formula of a tableau resulting in a contradiction can be proved in B.

If the initial formula α is:

$$(\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_n) \quad \beta$$

the tableau will be the following:

true	false
α_1	β
.	δ_1
.	.
.	.
α_n	.
γ_1	δ_p
.	
.	
.	
γ_m	

in which we must admit that one of the terms $\alpha_1,\ldots,\alpha_n$, $\gamma_1,\ldots,\gamma_m$ is identical with one of the terms $\beta,\delta_1,\ldots,\delta_p$. This tableau gives means of proving, by PC and the axiom A1[5], the following thesis:

$$\vdash_B (\alpha_1 \quad \ldots \quad \alpha_n \quad \beta) \quad (\alpha_1 \quad \ldots \quad \alpha_n \quad \gamma_1 \quad \ldots \quad \gamma_m \quad \beta \quad \delta_1 \ldots \quad \delta_p)$$

from which by contraposition we deduce,

$$\vdash_B \quad (\alpha_1 \quad \ldots \quad \alpha_n \quad \gamma_1 \quad \ldots \quad \gamma_m \quad \beta \quad \delta_1 \ldots \quad \delta_p) \quad (\alpha_1 \quad \ldots \quad \alpha_n \quad \beta).$$

Now the antecedent of this implication is deducible in B as tautological[6], since the expression in parentheses contains a conjunction of two contradictory terms. Thus, by detachment,

$$\vdash_B \quad (\alpha_1 \quad \ldots \quad \alpha_n \quad \beta)$$

i.e. the *initial formula*:

[5] In order to demonstrate the completeness of B, this intervention of A1 evidently is necessary only if the contradiction is already in W_0, since there is no other properly modal expression in the other worlds.

[6] R4, A1 and R3 permit to obtain every valid expression of PC and, therefore, every tautologically true expression.

$$\vdash_B (\alpha_1 \quad \ldots \quad \alpha_n) \quad \beta.$$

Let us now show that:

Every initial formula of a tableau can be deduced in B from the initial formula, supposed to have been demonstrated, of whatever tableau following from it.

Let us suppose that we have already demonstrated the *initial formula* of any tableau, other than the corresponding to the world origin:

$$\vdash_B (\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_n) \quad \beta.$$

Then

$$\vdash_B (L\alpha_1 \; L\alpha_2 \quad ..L\alpha_n) \quad L\beta$$

by R4, A2 and

$$\vdash_B (Lp_1 \; Lp_2 \ldots Lp_n) \quad L(p_1 \; p_2 \ldots p_n).$$

Let

$$(\alpha'_1 \quad \alpha'_2 \ldots \quad \alpha'_m) \quad \beta'$$

be the *initial formula* of the preceding tableau[7]. In this preceding tableau, $L\alpha_1, L\alpha_2, \ldots, L\alpha_n$ will appear on the left and $L\beta$ on the right. Then basing on PC and taking the axiom A1 we can prove that

$$\vdash_B (\alpha'_1 \quad \alpha'_2 \ldots \quad \alpha'_m \quad \beta') \quad (L\alpha_1 \; L\alpha_2 \ldots L\alpha_n \quad L\beta)$$

and, hence, by contraposition, that

$$\vdash_B [(L\alpha_1 \; L\alpha_2 \ldots L\alpha_n) \quad L\beta] \quad [(\alpha'_1 \quad \alpha'_2 \ldots \quad \alpha'_m) \quad \beta'].$$

Since the antecedent of the last formula has been proved, its consequent follows by detachment.

Till now we have not considered cases in which the procedure cannot determine the value of an expression without ambiguity and where we, consequently, are obliged to consider two or three distinct possibilities. In any such case, the expression:

$$\alpha_1 \quad \alpha_2 \quad \ldots \quad \beta_1 \quad \beta_2 \ldots$$

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[7] When the question is, as here, to demonstrate the completeness of B, the preceding tableau can only be the one corresponding to W_0, of which the initial formula is then reduced to "β'", i.e. to the thesis we have to demonstrate.

obtained as conjunction of the expressions of the left column and of the negations of expressions β_i of the right column is equivalent, by PC, to the disjunction of two or three expressions obtained by addition to the preceding conjunction of other conjuncts. Let us suppose, for example, that the expression:

$$\gamma \equiv \delta$$

is in the column of the false, then

$$\alpha_1 \quad \alpha_2 \quad \ldots \quad \beta_1 \quad \beta_2 \ldots \quad (\gamma \equiv \delta)$$

is equivalent to

$$(\alpha_1 \quad \alpha_2 \quad \ldots \quad \gamma \quad \beta_1 \quad \beta_2 \ldots \quad \delta) \quad (\alpha_1 \quad \alpha_2 \quad \ldots \quad \delta \quad \beta_1 \quad \beta_2 \ldots \quad \gamma).$$

The semantical procedure obliges us to consider separately each disjunct which has to lead to a contradiction if the expression is valid. But if each disjunct, D1, D2, possibly D3 leads, directly or indirectly, to a contradiction, our preceding demonstrations prove that it will be possible to deduce:

$$\vdash_B \quad D1$$
$$\vdash_B \quad D2$$
$$\vdash_B \quad D3$$

and, thereby,

$$\vdash_B \quad (D1 \quad D2 \quad D3)$$

expression equivalent (in the former notation) to:

$$\vdash_B \quad (\alpha_1 \quad \alpha_2 \quad \ldots \quad \beta_1 \quad \beta_2 \ldots).$$

Our preceding considerations show that from such a thesis the initial formula of the corresponding tableau can be easily deduced.

Let us first observe that this demonstration of the completeness of B by a few modifications can be adjusted to the demonstration of the completeness of M or T. We obtain the latter system by taking again the two axioms and the four rules on which we have founded B, under the proviso that the *rule of substitution* R1 is not restricted (by forbidding iteration of modality) and the *rule of necessitation* R4 is extended to the following form:

If the WFF α is valid in PC or is a thesis of T, then Lα is thesis of T.

To get a demonstration of the completeness of T it is sufficient to omit the notes 5 and 7 in the demonstration of completeness for B and to replace the sign "$\vdash_B$" by "$\vdash_T$".

Furthermore, since every thesis of S5 may be related to an expression of the first modal degree, the proof of the completeness of B can be used to demonstrate simply the completeness of S5. It will suffice to prove, in S5, the validity and deducibility of few equivalences needed for reducing expressions α of S5 to formulas α‘ of the first modal degree[8]. Next the completeness of S5 will be established by the following five successive inferences:

1) if α is S5-valid, α‘ is S5-valid;
2) if α‘ is S5-valid, α‘ is B-valid;
3) if α‘ is B-valid, α‘ is B-deducible (due to the completeness of B);
4) if α‘ is B-deducible, α‘ is, a fortiori, S5-deducible;
5) if α‘ is S5-deducible, α is S5-deducible (by recourse to the aforementioned equivalences).

Let us now consider a system BOD[9] founded on the three axioms:

A1 Op O p
A2 O(p q) (Op Oq)
A3 P’(p q) (P’p P’q)

and the following six rules:

R1 *Rule of substitution*, with the restriction that the substitution of a WFF in a thesis does not generate any iteration of modality

R2 *Rule of replacement*, with the classical definitions of PC and the definitions:

$Pp =_{df}$ O p
$Lp =_{df}$ p Op P’ p
$Mp =_{df}$ p O p P’p

R3 *Rule of detachment*

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[8] Cf. Hughes & Cresswell [5], p. 49-53.

[9] By these letters, BOD, I intend to suggest that this *basic* (B) system is together *ontic* (O) and *deontic* (D). I already outlined it in Gardies [2] and Gardies [3]. Here the main properties of BOD will be established.

R4 $\vDash_{PC} \alpha$ $\vdash_{BOD} \alpha$

R5 $\vDash_{PC} \alpha$ $\vdash_{BOD} O\alpha$

R6 $\vDash_{PC} \alpha$ $\vdash_{BOD} P' \quad \alpha$

From such an axiomatic system it is easy to deduce (using particularly the definition of "Lp") the two axioms and the four rules of the system B which is therefore included in the system BOD.

The semantics related to the system BOD is built on the following simple and intuitive principle: when we bring into play notions like *obligatory* and *permitted*, not only we have to take into consideration a set of *possible worlds* accessible to the first world W_0, but also to distinguish among the possible worlds two complementary sub-sets, first for the worlds acknowledged as (morally, juridically, etc.) *admissible* or *positive*, and the second for those which are acknowledged as *inadmissible* or *negative*. Thus we shall consider that α is *obligatory* in W_0, if and only if α is *true* in every admissible world. As regards the semantical characterization of the *permitted*, its already emphasized ambiguity allows to distinguish a *weak permission* (interdefinable with the preceding *obligation*) designated by P, and a *strong permission* designated by P'.

Thus we semantically interpret "Oα is true in W_0" as

α *is true in every positive world accessible to* W_0

and "Oα is false in W_0" into:

There is at least one positive world accessible to W_0 *in which* α *is false.*

The classical definition:

$Pp =_{df} \quad O \quad p$

will then allow to deduce the respective derived forms for "Pα is false in W_0":

α *is false in every positive world accessible to* W_0

and for "Pα is true in W_0":

There is at least one positive world accessible to W_0 *in which* α *is true.*

For the *strong permission*[10], "P'α is true in W_0" may be characterized by:

α *is false in every negative world accessible to* W_0

and "P'α is false in W_0" by:

There is at least a negative world accessible to W_0 *in which* α *is true*.

In order that such a semantics could plainly correspond with the deontic functors we aim to characterize, the origin world W_0 should be neither positive nor negative, and its accessibility to itself is out of the question; and since the worlds W_{+1}, W_{+2},..., W_{-1}, W_{-2},..., either positive or negative, accessible to it, contain no other modal expression (as the WFF of BOD, like those of B, have no iterated modality), there is no place for reflexivity, symmetry or transitivity of the relation of accessibility.

Furthermore we cannot forget that, if we identify the *positive* with *what one has the right to do*, between an action and its abstention, at least one of both must be under some circumstances admissible, so that among all possible worlds between which our freedom can choose, there is at least one which is *positive*. Thus we are forced to assume that *in every case* there is always at least one positive world W_{+i} accessible to the first world W_0[11].

It can be easily shown, under these conditions, that the three axioms of BOD are valid, because they cannot be falsified, and that the six rules of the system are preserving validity. Thus all theses of BOD will be valid.

For the purpose of proving the completeness of BOD we shall first eliminate from the expressions to validate all modal operator other than those we have chosen as *undefined first terms*, i.e. O and P'[12]. We have characterized as valid in BOD every expression of the system which, if assumed false in W_0, would lead to a contradiction (i.e. to the

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[10] Such a semantics of the *strong permission* can already be found in Bailhache [1].

[11] If not, it would be impossible to validate the axiom A1.

[12] It may be remarked that we abstain from introducing into BOD an obligation O' which would be interdefinable with P' under the form:

$O'p =_{df}$ P' p

I have discussed the question of this abstention in Gardies [2], pp. 193-194: such a functor seems to have no corresponding term in most vernacular languages.

presence of the same expression in the true and in the false) either in the first world W_0 itself, or in a positive world W_{+i}, or in a negative world W_{-i}. I shall successively show for each of these three cases that the expression for which the apagogic procedure thus leads to a contradiction is deducible in BOD.

Let us first examine the case where a contradiction appears in the world origin W_0. Then the thesis is already valid by virtue of the sole properties of PC. R1, R2 and R4 are thus sufficient for its deduction in BOD.

Let us then consider the case of a contradiction in a positive world W_{+i}. Here there are two possibilities, either we have been led to open the world W_{+i} because an expression O_β was false in W_0; or there was no expression O_β false in W_0, but only n expressions $O\alpha_1$, $O\alpha_2$,...$O\alpha_n$ ($n \geq 1$) all true in W_0, and we have opened the world W_{+i} only because our semantical procedure stipulated that, in every case, there was at least one positive world accessible to W_0.

Let us then consider the first possibility of our second case. We can again take the expression of *initial formula*, such as was used for the completeness proof of the system B. The initial formula will thus also be:

$$(\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n) \quad \beta$$

β being the expression initially placed on the right of the tableau, $\alpha_1, \alpha_2, \dots \alpha_n$ being the n expressions initially placed on the left. We shall show that this initial formula is deducible in BOD exactly in the same way as for B[13]. Let us then suppose that we have proved the initial formula of a tableau W_{+i} whatever:

$$\vdash_{BOD} (\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n) \quad \beta.$$

We can then prove:

$$\vdash_{BOD} (O\alpha_1 \; O\alpha_2 \quad ..O\alpha_n) \quad O\beta.$$

by successive recourse to R5, to A2 and to

$$\vdash_{BOD} (Op_1 \; Op_2 \quad ..Op_n) \quad O(p_1 \; p_2 \quad ..p_n).$$

Let us now designate by "β'" the thesis to prove which appears on the right of the tableau of W_0. In this tableau $O\alpha_1, O\alpha_2, \dots, O\alpha_n$ will appear

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[13] Cf. above the demonstration of the completeness of B, with which the following demonstration can be compared.

on the left, and Oβ on the right. Then basing upon PC we may show that

$$\vdash_{BOD} \quad \beta' \quad (O\alpha_1 \; O\alpha_2 \; ..O\alpha_n \quad O\beta)$$

and hence, by contraposition:

$$\vdash_{BOD} [(O\alpha_1 \; O\alpha_2 \; ..O\alpha_n) \quad O\beta] \quad \beta'$$

implication from which we can detach β', since the antecedent is already proved.

Now, we pass to the case, where there are some expressions commanded by the functor O only in the column of the true of W_0. If there are n $O\alpha_i$ in this column of the true, we have (β' being the thesis to prove):

$$\vDash_{PC} \quad \beta' \quad (O\alpha_1 \; O\alpha_2 \; ..O\alpha_n)^{14}$$

The presence of formulas $O\alpha_i$ true in W_0 will have obliged us to open a positive world, W_{+i} represented by a tableau of the form:

true	false
α_1	β_1
.	.
.	.
.	.
α_n	β_p
γ_1	
.	
.	
.	
γ_m	

in which one of the terms $\alpha_1,...\alpha_n,\gamma_1,...\gamma_m$ is identical with one of $\beta_1,...,\beta_p$. Starting from this tableau, we obtain:

$$\vDash_{PC} (\alpha_1 \; \alpha_2 \; ... \; \alpha_n) \quad (\alpha_1 \; ... \; \alpha_n \; \gamma_1 \; ... \; \gamma_m \quad \beta_1 \; ... \quad \beta_p)$$

in which one of the α_i or γ_i is identical with some β_i. We can then prove

$$\vdash_{BOD} (O\alpha_1 \; O\alpha_2 \; ..O\alpha_n) \quad (O\alpha_1 \; ..O\alpha_n \; O\gamma_1 \; ..O\gamma_m \; O \quad \beta_1 \; ..O \quad \beta_p)$$

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14 A PC-valid expression, by R4, BOD-deducible.

by successive recourse to R5, A2 and to

$$\vdash_{BOD} (Op_1\ Op_2\ \ ..Op_n) \equiv O(p_1\ p_2\ \ ..p_n).$$

Let β_j be the expression identical with one of the α_i's or γ_i's. From the thesis equivalent to A1:

$$\vdash_{BOD}\ (Op\ \ O\ \ p)$$

we can deduce:

$$\vdash_{BOD}\ (O\beta_j\ O\ \ \beta_j).$$

Now

$$\vDash_{PC}\ (O\beta_j\ O\ \ \beta_j)\ \ \ (O\alpha_1\ \ ..O\alpha_n\ O\gamma_1\ \ ..O\gamma_m\ O\ \ \beta_1\ \ ..O\ \ \beta_p).$$

Both latter theses, together with the following contraposition of another thesis previously established:

$$\vdash_{BOD}\ (O\alpha_1\ \ ..O\alpha_n\ O\gamma_1\ \ ..O\gamma_m\ O\ \ \beta_1\ \ ..O\ \ \beta_p)\ \ \ (O\alpha_1\ O\alpha_2\ \ ..O\alpha_n)$$

allow us to deduce:

$$\vdash_{BOD}\ (O\alpha_1\ O\alpha_2\ \ ..O\alpha_n).$$

Relating this result to the contraposition of the PC-valid expression considered at the beginning:

$$\vDash_{PC}\ (O\alpha_1\ O\alpha_2\ \ ..O\alpha_n)\ \ \ \beta'$$

we get the thesis to demonstrate, i.e. β'.

Let us finally examine the third case when a contradiction appears in a negative world W_{-i}. We shall here take again the expression of *initial formula*, however, not forgetting that its form will not be the same as before. Thus, we consider

$$\alpha\ \ \ (\beta_1\ \ \ \beta_2\ \ \dots\ \ \beta_n)$$

with α being the unique expression initially introduced on the left side of the tableau W_{-i} and β_1, β_2,...,β_n the n (n≥0) expressions initially added on the right side of the tableau. We may represent it as

true	false
α	β_1
γ_1	.
.	.
.	.
.	β_n
γ_m	δ_1
	.
	.
	.
	δ_p

where one of the terms $\alpha,\gamma_1,\ldots,\gamma_m$ is identical with one of $\beta_1,\ldots,\beta_n,\delta_1,\ldots,\delta_p$. The above tableau allows to establish

$$\vDash_{PC} (\alpha \quad \beta_1 \ldots \beta_n) \quad (\alpha \; \gamma_1 \ldots \gamma_m \; \beta_1 \ldots \beta_n \; \delta_1 \ldots \delta_p)$$

an expression in which one of the terms $\alpha,\gamma_1,\ldots,\gamma_m$ is identical with one of the terms $\beta_1,\ldots,\beta_n,\delta_1,\ldots,\delta_p$. Thus, we also have

$$\vDash_{PC} \quad (\alpha \; \gamma_1 \ldots \gamma_m \; \beta_1 \ldots \beta_n \; \delta_1 \ldots \delta_p)$$

These two expressions yield

$$\vDash_{PC} \quad (\alpha \quad \beta_1 \ldots \beta_n)$$

which is equivalent to our *initial formula*. From this by R6 we deduce

$$\vdash_{BOD} P' \quad [\alpha \quad (\beta_1 \quad \beta_2 \quad \ldots \quad \beta_n)]$$

i.e.

$$\vdash_{BOD} P'[\alpha \quad (\beta_1 \; \beta_2 \quad \ldots \quad \beta_n)]$$

and, by A3,

$$\vdash_{BOD} P'(\beta_1 \quad \beta_2 \quad \ldots \quad \beta_n) \quad P'\alpha$$

Subsequently, due to

$$\vdash_{BOD} (P'p_1 \; P'p_2 \; ..P'p_n) \quad P'(p_1 \quad p_2 \quad \ldots \; p_n)$$

we get

$$\vdash_{BOD} (P'\beta_1 \; P'\beta_2 \; ..P'\beta_n) \quad P'\alpha.$$

Now, in W_0, P'α is in the column of the false, P'β_1, P'β_2,…,P'β_n are on the side of the true. Therefore, if β' is a thesis to prove, we have:

$$\vDash_{PC} \quad \beta' \quad (P'\beta_1\ P'\beta_2\ \ ..P'\beta_\nu.\ \ P'\alpha)$$

or, by contraposition,

$$\vDash_{PC} [(P'\beta_1\ P'\beta_2\ \ ..P'\beta_n) \quad P'\alpha] \quad \beta'$$

which allows to deduce "β'".

Summing up, our completeness proof of BOD proceeded by the three following cases:

C1) when the contradiction is already in W_0;
C2) when a contradiction is in a W_{+i}; this case may be divided into two possibilities, C21, where the opening of the positive world proceeded from the falsehood of an expression Oβ in W_0, and C22, where this positive world was opened although there was no such false expression in W_0;
C3) when a contradiction is in a negative world W_{-i}.

The present demonstration should only be completed in the situations where the procedure of validation does not allow to determine, without any ambiguity, the truthvalue of a given expression and then we should be obliged to consider successively two or three distinct possibilities. I shall here merely refer to the way of elimination of that difficulty at the end of our demonstration for the system B[15].

We shall now consider a system BD included in BOD whose characteristic feature is that the only used modal functors are the deontic functors, O and P. Its axiomatics will be constituted by A1, A2 and the rules R1, R3, R4, R5 of BOD, taken again without any modification, and the *rule of replacement* R2 with the classical definitions of PC and the modal definition:

$$Pp =_{df} \quad O \ \ p.$$

BD is similar with the system D of Hanson [5], again used by Bailhache [1], with this important difference[16] that BD, by the rules of good formation inherited from B, contain only theses of D without iteration of

[15] Cf. above.
[16] This we had already underlined in Gardies [2], p. 195.

modalities. The semantical characteristics for Oα and, thereby, for Pα will be the same as in the case of BOD[17].

It can easily be verified that the axioms of BD are valid and that the rules of the system are validity preserving. In order to prove completeness of BD, it is sufficient to apply to BD our reasoning concerning the cases C1 and C2 (the only that can occur here without any mention of any negative world) of the demonstration of the completeness of BOD.

As for BOD, it should be stressed that the reference to *strong* and *weak* permission, characterized in particular by the equivalences:

P’(p q) ≡ (P’p P’q)
P (p q) ≡ (Pp Pq),

due to a tradition originated by G. H. von Wright[18], must not be understood with the properly logical meaning of the words *strong* and *weak*. Indeed, none of them is stronger than the other, since none implies the other as it can be easily verified.

We could speak of *logically stronger* permission only if we would definitionally introduce in BOD new deontic functors with the *definiens* being for instance one of the following[19]:

Pp P’p
Pp P’p
Pp P’p

Then it is easy to understand that the first would be *logically stronger* than P and P’, the second would be *logically stronger* than P and the third than P’. It is probable that the first is effectively felt to be deontically stronger than P and than P’; on the other hand, it is hardly credible that the second and the third were felt to be deontically stronger than P and than P’, respectively; on the contrary the negation in these formulas seems rather to suggest a restriction[20].

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[17] Cf. above.

[18] Cf. von Wright [7].

[19] To make the explanations more intuitive, I prefer to use the defined term P than the primitive term O.

[20] If we compare the third *definiens* proposed here with the definition L in BOD, we shall remark that it signifies an impossibility of p, which only would be limited to the set of the positive or negative worlds, with the exclusion of the world W_0 itself, in which p could still be true.

Let us summarize our results. We started from a basic modal system B and built two other systems, BD and BOD, limited to the first modal degree. For each of these systems we provided an axiomatics and a semantics and thus we proved their soundness and completeness. Besides it can be noticed that our demonstrations of completeness have always given means of deducing every expression whose procedure of validation could establish the validity. B and BD have no other common thesis than the theses of PC. If we consider the first as *basic ontic system*, for the reasons put forward by Pollock, we can consider the second as *basic deontic system*. As far as BOD is concerned it seems that it can be considered as a *basic ontico-deontic system*, since not only every modal iteration is excluded from it, but it also contains every thesis of B and every thesis of BD, and contains no other strictly ontic theses than those of B, and no other strictly deontic thesis[21] than those of BD.

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[21] We qualify here as *strictly ontic* and *strictly deontic* every thesis which can be written without recourse to any other modal functors than, respectively:

- L and M,
- O and P.