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HISTORY AND DEVELOPMENT OF THE DISCURSIVE LOGIC*

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Abstract

In 1948, Stanisław Jaśkowski defined a logical system $\boldsymbol{D_2}$ of a discursive[1] sentential calculus. The aim of this paper is to introduce the reader to the basic ideas of the discursive logic and to show, in a historical perspective, its development originating from the two germ papers [19] and [20]. We intend to present some problems connected with it and outline the solutions they have received up to the present day.

Introduction

The paper consists of five parts, each devoted to one aspect of the discursive logic. To start with, we consider $\boldsymbol{D_2}$ and recall Jaśkowski's motivations, definitions and theorems. The system $\boldsymbol{D_2}$ is originally based on Lewis' system $\boldsymbol{S_5}$. Since the possibility symbol has different meanings in different modal systems, it is then reasonable to inquire whether $\boldsymbol{S_5}$ is the only system that may serve as the base for $\boldsymbol{D_2}$. This question will be answered in the second and fourth parts of this paper. Next question is to present a direct axiomatization of $\boldsymbol{D_2}$. We outline this problem in the third part. An algebraic semantics for the discursive system is discussed in the fifth part.

The standard logical notation is applied: the symbols , , , $\rightarrow$, $\leftrightarrow$, $\Box$ and $\Diamond$ denote negation, conjunction, disjunction, material implication, equivalence, necessity and possibility, respectively. We enrich the set of propositional connectives with three additional symbols ($_d$, $\rightarrow_d$,

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[1] To call the system, some authors used the adjective *discussive* what seemed to be a too literal translation from Polish into English. We have not decided to use the adjective.

↔ d).[2] Greek letters will denote formulae and p, q, r ..., propositional variables. We regard the logical systems as subsets of the set of *For* of formulae.

In the second part, some definitions will be treated in a uniform way. Generally, the remarks will function as lemmas or commentaries. The accompanying bibliography helps the reader to deepen his knowledge on the subject. We would like the reader to distinguish clearly between $\boldsymbol{D_2}$ and $\boldsymbol{J}$. Jaśkowski defined $\boldsymbol{D_2}$ by means of an interpretation in $\boldsymbol{S_5}$, but he did not show that $\boldsymbol{D_2}$ is a finitely axiomatizable system nor he gave any infinite axiom set for $\boldsymbol{D_2}$. In 1968 Newton C.A. da Costa and Lech Dubikajtis made a clear distinction between the systems, which turned out to be a breakthrough. For this reason, the symbol $\boldsymbol{D_2}$ represents Jaśkowski's system and so does a system in which there are only discursive connectives. The symbol $\boldsymbol{J}$ stands for a discursive system expressed in a modal language.

For the purpose of the present text the following two properties are of a special interest: an *inconsistent* system and *overfilled* system.

Definition 1. Let S be a formal system. A system S is called inconsistent if there is at least one formula such that this formula as well as its negation are theses of S.

Definition 2. Let S be a formal system. A system S is called overfilled (trivial, over-complete) if each well-formed expression is its thesis.[3]

1. The beginnings

There were at least two reasons to construct $\boldsymbol{D_2}$:

(i) vague terms and imprecise concepts. It is common in everyday life as well as in science to make use of concepts (terms) which are more or less imprecise (vague). Each imprecision (vagueness) of a concept (term) may lead to apparent contradictions;

(ii) empirical hypotheses. Scientists cannot at times explain the results of an experiment by means of a coherent theory. They are used to explaining it by distinct *working* hypotheses.[4]

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[2] See § 1.3.

[3] [27], p. 149, see also [42], p. 110.

[4] See [19], p. 144.

1.1. Jaśkowski's logical inspirations

Jaśkowski wished to construct a logical calculus which: (i) when applied to the contradictory systems would not always entail their over-completeness; (ii) would be rich enough to enable practical inference; (iii) would have an intuitive justification[5]. The issue how to find such a system makes Jaśkowski consider the known solutions.

1.1.1. Kolmogorov's system

The choice of Jaśkowski does not appear to be accidental and may be explained by his interest in the intuitionistic logic[6]. Ultimately he confined himself to examining Kolmogorov's third[7] system of axioms composed of the implicational part of Hilbert's positive logic:

(Ax.1) $\alpha \rightarrow (\beta \rightarrow \alpha)$,
(Ax.2) $(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta)$,
(Ax.3) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$,
(Ax.4) $(\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

and the additional schema:

(Ax.5) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \quad \beta) \rightarrow \quad \alpha)$

The system is closed on the detachment rule[8].

Remark 1. In Kolmogorov's system: (F1) $\alpha \rightarrow (\quad \alpha \rightarrow \beta)$ is not valid, but (F2) $\alpha \rightarrow (\quad \alpha \rightarrow \quad \beta)$ is valid.[9]

1.1.2. Lewis' system of strict implication. Many-valued logics

Both have been rejected yielding no satisfactory results: the former due to the insufficiency of theses for practical inference[10]; the latter because of a lack of *a publication related directly to the problem*[11].

1.2. Concept of discursive system and discursive assertion

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[5] (i)-(iii) are the literal quotations from [19], p. 145.

[6] See e.g. Zawirski, Z., *Geneza i rozwój logiki intuicjonistycznej*, Kwartalnik Filozoficzny 16(2).

[7] See [22], pp. 81-82.

[8] We omit (Sub) as irrelevant to further analysis.

[9] [19], p. 147.

[10] See (ii), § 1.1.

[11] [19], p. 147.

By *discursive system,* we understand a system that does not exclude directly inconsistent sentences (hypotheses, formulae)[12]. The generic example provided by Jaśkowski runs as follows. Consider a group of people discussing on a topic. Assume that terms and statements they use are vague. It would not be rash to say that if all the statements were put together to form a system we would be made to accept *a seeming inconsistency*[13]. We need not be surprised that some provision is needed to bring out the nature of such statements. Jaśkowski suggested that it should be proper to precede each statement (thesis) by the reservation: *in accordance with the opinion of one of the participants in the discourse* or *for a certain admissible meaning of the terms used*[14]. For on this reservation he had to distinguish a connective that has the properties. Therefore a concept of *discursive assertion* was given.

It is a truism to say that discursive assertion has little in common with the classical one. The truth-functional character of classical assertion *(a)* is made apparent in the following truth table:

α	$(a)\alpha$
0	0
1	1

Classical assertion corresponds to all the expressions of a natural language whose aim is to reinforce an utterance. You say: *I went to hospital yesterday*, but you can also say: *I did go to hospital yesterday* to give emphasis to your utterance[15]. Discursive assertion implicitly includes the reservation *in accordance with ...* That is why, it must not be conceived as an extensional connective and it corresponds rather to the modal connective of possibility ($\Diamond$). Stanisław Jaśkowski describing the character of discursive assertion referred to logical and partially philosophical intuitions. In fact, a problem of philosophical importance arises concerning certain statements, such as *2 + 2 = 4*, which have al-

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[12] Comp. *Definition 1*.

[13] *Seeming inconsistency* should be differed from *overcompleteness.* A discourse is *overfilled* (*overcomplete*) if the opinion (opinions) of one of the participants is (are) self-contradictory. It finds expression in the formula : (F) $(\alpha \quad \alpha) \to \beta$.

[14] [19], p. 149.

[15] From a grammatical point of view, we only emphasise the main verb of a sentence.

ways been true. Their validity does not depend on *the opinion of one of the participants in the discourse*. To emphasise the nature of the statements it is important to precede each of them by the reservation: *It is necessary that* ... and we do not make a mistake to identify the reservation with the necessity symbol[16].

1.3. Problem with MP[17]*. Discursive connectives*

An attempt to base a discursive system upon ordinary two-valued logic fails. It is apparent when we take into account the detachment rule: if we consider two premises:

(i) $\alpha \rightarrow \beta$,
(ii) α

having the following meanings:

(i) it is possible that if α, then β,
(ii) it is possible that α

we find it faulty to accept the conclusion:

(iii) β (it is possible that β).

It is due to the fact that ***S5*** is not closed on the rule:

(R*) $\Diamond\alpha, \Diamond(\alpha \rightarrow \beta) / \Diamond\beta$[18].

The best thing one can do under this confusing circumstance is to choose a logical connective that has the similar properties as implication, but it does not cause a discursive system to collapse. Hence, Jaśkowski introduced a new connective (*discursive implication*):

*Definition 3***.** $p \rightarrow_d q = \Diamond p \rightarrow \ q$.

Although the definition was chosen to show the effects of natural language[19], the deciding factor was to remove Duns Scotus' thesis (S) $\alpha \rightarrow (\ \ \alpha \rightarrow \beta)$from our system[20]. For if the definition:

[16] See §§ 2.2. and 2.3.
[17] *MP* (Modus Ponens) stands for the detachment rule.
[18] Comp. [9], p. 9.

(N-Df.) $p \rightarrow_{d} q = \Diamond p \rightarrow \Diamond q$[21]

was accepted we were forced to accept the formula:

(F) $(p \rightarrow_{d} q) \rightarrow_{d} ((p \rightarrow_{d} q) \rightarrow_{d} r)$

as a thesis. Moreover, *Definition 3* enables us to avoid the difficulties with (MP). Now we have:

(MP)* $\alpha, \alpha \rightarrow_{d} \beta / \beta.$

Discursive equivalence ($\leftrightarrow_{d}$) was similarly expressed:

*Definition 4***.** $p \leftrightarrow_{d} q = (\Diamond p \rightarrow q) \quad (\Diamond q \rightarrow \Diamond p).$

In analogy with *Definition 3*, the following definition was rejected as inappropriate:

(N-Df.) $p \leftrightarrow_{d} q = (\Diamond p \rightarrow \Diamond q) \quad (\Diamond q \rightarrow \Diamond p)$[22].

Remark 2. Discursive equivalence is not symmetric[23].

Enriching the system with an additional connective of the *discursive conjunction*[24]:

*Definition 5***.** $p \quad_{d} q = p \quad \Diamond q$[25],

it makes possible to simplify the *discursive equivalence* to:

Definition 4'. $p \leftrightarrow_{d} q = (p \rightarrow_{d} q) \quad_{d} (q \rightarrow_{d} p)$

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[19] See § 1.2.

[20] Comp. *Remark 1.*

[21] Comp. [17], p. 42 (Definition 4).

[22] It would straight lead to accept (F) $(p \leftrightarrow_{d} q) \rightarrow_{d} ((p \leftrightarrow_{d} q) \rightarrow_{d} r)$ as a thesis.

[23] See § 5.4.

[24] It is worth mentioning that the authors of [12] considered three discursive systems: ***J*** (see § 2.1.1.), ***D2*** (without discursive conjunction) and ***D2'*** (with discursive conjunction). See *op. cit*., pp. 552-554.

[25] Jaśkowski gave a concept of *discursive conjunction* in [20].

or to:

Definition 4''. $p \leftrightarrow_{\mathrm{d}} q = (\Diamond p \to q) \quad \Diamond(\Diamond q \to p)$.

Remark 3. *Definitions 4*, *4'* and *4''* are equivalent.

*1.4. Discursive system **D2***

To begin with, we are interested in comparing the languages of ***D2*** and of ***S5*** to provide a definition of the former by an interpretation in the latter.

(i) $\boldsymbol{L_{D2}} = \langle For_{D2}, \ , \ , \ _{\mathrm{d}}, \to_{\mathrm{d}} \rangle$,
(ii) $\boldsymbol{L_{S5}} = \langle For_{S5}, \ , \ , \ , \to, \Diamond, \Box \rangle$.

Definition 6. Let f be a function of $\boldsymbol{L_{D2}}$ into $\boldsymbol{L_{S5}}$ (f : For_{D2} For_{S5}) defined as follows:

(i) $f(p) = p$, p is a sentential variable,
(ii) $f(\ \alpha) = \ f(\alpha)$,
(iii) $f(\alpha \ \beta) = f(\alpha) \ f(\beta)$,
(iv) $f(\alpha \ _{\mathrm{d}} \beta) = f(\alpha) \ \Diamond f(\beta)$,
(v) $f(\alpha \to_{\mathrm{d}} \beta) = \Diamond f(\alpha) \to f(\beta)$.

Definition 7. $\alpha \ For_{D2}$: $\alpha \ \boldsymbol{D_2}$ iff $\Diamond f(\alpha) \ \boldsymbol{S_5}$.

Remark 4. ***D2*** is decidable since ***S5*** is decidable.

1.5. Metalogical theorems

The following three theorems of Jaśkowski determine the properties of ***D2***:

Theorem 1. Each thesis α of ***CPC***[26] containing besides sentential variables no other signs but , , $\to$ and $\leftrightarrow$ is transformed into a thesis β of ***D2*** after replacing in α the connectives , , $\to$, $\leftrightarrow$ by , $_{\mathrm{d}}$, $\to_{\mathrm{d}}$, $\leftrightarrow_{\mathrm{d}}$, respectively.

Theorem 2. If α is a thesis of ***CPC*** containing only variables and the connectives and , then: (i) α and (ii) $\alpha \to_{\mathrm{d}} q$ are the theses of ***D2***.

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[26] *CPC* stands for classical propositional calculus.

Theorem 3. Each thesis β of $\boldsymbol{D_2}$ is transformed into a thesis α of ***CPC*** by replacing in β the connectives , d, →d, ↔d with , , →, ↔, respectively[27].

The purport of *Theorems 1*, *2* and *3* is of vital importance. A *simple replacement* (not interpretation (*Definitions 6* and *7*)) of the connectives demonstrates that:

(i) $\boldsymbol{D_2}$ contains ***CPC+***;
(ii) $\boldsymbol{D_2}$ contains the set of all disjunctio-negational theses of ***CPC***;
(iii) $\boldsymbol{D_2}$ **+** (after the transformation) may be identified with ***CPC+***;

where ***CPC+*** ($\boldsymbol{D_2}$**+**) is an abbreviation for the positive part of ***CPC*** ($\boldsymbol{D_2}$).

2. Axiomatization of *J*

The year 1967 was a turning point in the development of the discursive logic. Newton C.A. da Costa and Lech Dubikajtis met in Paris and gradually commenced developing of the logic ever since[28]. Their joint paper[29] appeared in 1968 and dealt with the first axiomatics for $\boldsymbol{D_2}$ and ***J***. A number of publications have followed with the course of time.

We will present a number of different axiom systems for the discursive logic, however, in changed notation, for the sake of clarity and precision of this paper. The first system will be denoted as ***J*** and the subsequent systems by $\boldsymbol{J_1}$, $\boldsymbol{J_2}$, $\boldsymbol{J_3}$, ... etc. Let us give a definition of ***J***:

Definition 8. $\boldsymbol{J} = \{\alpha \quad For_{S5} : \Diamond\alpha \quad \boldsymbol{S_5}\}$.

2.1. First axiomatics for J

The authors of [12] use the standard logical symbols. Actually, the only difference is that they used small Latin letters as predicates.

Definition 9. Let *P* be a set of theorems of homogeneous 1-argument predicate calculus (*HPC*) that is for any β *P* only one individual vari-

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27 To analyse the proofs of the theorems, see [19], pp. 151-154 and [27], pp. 152-154.

28 Compare with [3], p. 116.

29 Namely [12].

able may occur in β (n-times, n = 1,2,3...). Now let Q be a set of expressions obtained from P by omission of all the individual variables.

Remark 5. Together with the omission of an individual variable, e.g. x, we omit all the parentheses accompanied with the variable.

Example. Let β be the form:
β : $_x(p(x) \rightarrow q(x)) \rightarrow (\; _x p(x) \rightarrow \; _x q(x))$.
Notice β P. Now we reduce β to the expression α of the form:
α : $(p \rightarrow q) \rightarrow (\; p \rightarrow \; q)$.

Remark 6. It was first proved by Wajsberg[30] that the quantifiers (existential) and (universal) correspond to the modal connectives $\Diamond$ and $\Box$, respectively. What is more, if we consider the above it is easy to realise that the system Q may be treated as the modal system ***S5***.

Now let us define the discursive connectives:

(R1) $\alpha \rightarrow_d \beta = \; \alpha \rightarrow \beta$,
(R2) $\alpha \leftrightarrow_d \beta = (\alpha \rightarrow_d \beta) \; (\beta \rightarrow_d \; \alpha)$,
(R3) $\alpha \; _d \beta = \alpha \; \beta$,
(RD) = {(R1), (R2), (R3)}.

Needless to say, (RD) is very similar to Jaśkowski's definitions[31]. The authors of [12] also examine the rules (MP)* and (R4) α / α.

Remark 7. Every truth-value function may be defined by and [32].

From *Remark 7* and (RD), it results that:

(R5) $\alpha \; \beta = \; (\; \alpha \; \beta)$,
(R6) $\alpha \rightarrow \beta = \; \alpha \; \beta$,
(R7) $\alpha \leftrightarrow \beta = (\alpha \; \beta) \; (\; \alpha \; \beta)$,
(R8) $\alpha = \alpha \leftrightarrow_d (\alpha \; \alpha)$,
(R9) $\alpha = \; \alpha$,
(RW) = {(R5)-(R9)}.

[30] Comp. [19], pp. 148-149.
[31] See § 1.3.
[32] Comp. [33], p. 26.

Replace (R4) with (Ax) $(\alpha \leftrightarrow_d (\alpha \quad \alpha)) \to_d \alpha$ in virtue of the rules (MP)* and (R8). (R_D) can be modified to the form:

(R1)' $\alpha \to_d \beta = (\alpha \leftrightarrow_d (\alpha \quad \alpha)) \quad \beta$,
(R2)' $\alpha \leftrightarrow_d \beta = ((\alpha \to_d \beta) \quad (\beta \to_d (\alpha \leftrightarrow_d (\alpha \quad \alpha)))$),
(R3)' $\alpha \quad _d \beta = (\alpha \quad (\beta \leftrightarrow_d (\beta \quad \beta))$),
(R_D)'= {(R1)', (R2)', (R3)'}.

Let *Q'* be a set obtained from *Q* by (R_W) and containing no other logical connectives but , , $\leftrightarrow_d$.

Theorem 4. Axioms of *Q'*, (Ax), (MP)*, (R_D)' and (R_W) constitute the complete axiomatization of ***J***.

To prove the completeness theorem the authors had recourse to an algebraic semantics for ***J***[33].

2.2. Systems J_1 and J_2[34].

Both of them were originally denoted by the letter *J*[35]. The paper dealing with the systems was published in 1975.

Definition 10. Let J_1 be a system defined on the language L_{S5} by the following axiom schemata and rules of inference:

(Ax.1) $\Box\alpha$ whenever α is a tautology of ***CPC***,
(Ax.2) $\Box(\Box(\alpha \to \beta) \to \Box(\Box\alpha \to \Box\beta))$,
(Ax.3) $\Box(\Box\alpha \to \alpha)$,
(Ax.4) $\Box(\alpha \to \Box\Diamond\alpha)$,
(Ax.5) $\Box(\Box\alpha \to \Box\Box\alpha)$,
(R1) $\alpha, \Box(\alpha \to \beta) / \beta$,
(R2) $\Diamond\alpha / \alpha$[36].

Remark 8. The role of axiom schemata of S_5 can be taken on by the following:

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[33] Comp. § 5.1., more details in [12], p. 555.
[34] We do not put the axiomatizations of the discursive logic in chronological order.
[35] The symbols J_1 and J_2 have their origin in our convention.
[36] Compare with J_3 (see § 2.3.).

(i) each tautology of ***CPC***,
(ii) $\Box(\alpha \rightarrow \beta) \rightarrow \Box(\Box\alpha \rightarrow \Box\beta)$,
(iii) $\Box\alpha \rightarrow \alpha$,
(iv) $\alpha \rightarrow \Box\Diamond\alpha$

and the rules (MP) and (RG) $\alpha / \Box\alpha$.

Remark 9. The formulae:

(F1) $\Box\alpha \rightarrow \Box\Box\alpha$,
(F2) $\Diamond\Diamond\alpha \rightarrow \Diamond\alpha$,
(F3) $\alpha \rightarrow \Diamond\alpha$,

are valid in $\boldsymbol{S_5}$.

Remark 10. The rules of inference:

$(R_W)'$ $\Box\alpha / \alpha$,
$(R_W)''$ $\Box\alpha, \Box(\alpha \rightarrow \beta) / \Box\beta$,
$(R_W)'''$ $\Box\alpha / \Box\Box\alpha$,

hold in $\boldsymbol{J_1}$.

Theorem 5. $\boldsymbol{J} = \boldsymbol{J_1}$.

Proof. We have to show that:

(i) $\boldsymbol{J}$ $\boldsymbol{J_1}$,
(ii) $\boldsymbol{J_1}$ $\boldsymbol{J}$.

(i) Assume that δ $\boldsymbol{J_1}$. Let A be a set of axiom schemata of $\boldsymbol{J_1}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n, \delta$ be a proof of δ by means of A, (R1) and (R2). Notice that $\Diamond\gamma_1, \Diamond\gamma_2 \ldots \Diamond\gamma_n, \Diamond\delta$ is a proof of $\Diamond\delta$ by $\Diamond A$, $\Diamond$(R1) and $\Diamond$(R2). Now, it suffices to show that $\Diamond A$, $\Diamond$ (R1) and $\Diamond$ (R2) are the rules of $\boldsymbol{S_5}$. Notice that $A = \Box A'$, where A' and the rules (MP), (RG) constitute a complete axiomatization of $\boldsymbol{S_5}$ (due to *Remark 8*). Now apply (RG) to A' to obtain A (the set of axiom schemata for $\boldsymbol{J_1}$). Make use of (F3) $\Box\alpha \rightarrow \Diamond\Box\alpha$ and gradually replace α by each of the axiom schemata. Apply (MP) to obtain that $\Diamond A$ $\boldsymbol{S_5}$. Therefore, we have $\Diamond A$ $\boldsymbol{S_5}$. It is easy to show that $\Diamond$(R1) $\boldsymbol{S_5}$ and $\Diamond$(R2) $\boldsymbol{S_5}$.

(ii) Assume that $\Diamond\delta$ $\boldsymbol{S_5}$ and the sequence $\gamma_1, \gamma_2 \ldots \gamma_n$, is a proof of $\Diamond\delta$ in $\boldsymbol{S_5}$. Now let us show that the following sequence: $\Box\gamma_1, \Box\gamma_2, \ldots, \Box\gamma_n, \Diamond\delta, \delta$ is a proof of δ on the ground of $\boldsymbol{J_1}$. If γ_i is a direct consequence of the preceding formulae γ_j and γ_k by virtue of (MP) (the system $\boldsymbol{S_5}$), then $\Box\gamma_i$ is obtained from $\Box\gamma_j$ and $\Box\gamma_k = \Box(\gamma_i \to \gamma_j)$ by $(R_w)''$ (the system $\boldsymbol{J_1}$). If γ_i arises from a preceding formula γ_k by means of (RG) (the system $\boldsymbol{S_5}$), then $\Box\gamma_i$ arises from $\Box\gamma_k$ by $(R_w)'''$ (the system $\boldsymbol{J_1}$). Since $\Box(\Box\Diamond\delta \to \Diamond\delta)$ $\boldsymbol{J_1}$ ((Ax3): $\Diamond\delta / \alpha$) and $\Box\Diamond\delta$ $\boldsymbol{J_1}$ ($\gamma_n = \Diamond\delta$) we have $\Diamond\delta$ $\boldsymbol{J_1}$ by (R1) and, consequently, δ $\boldsymbol{J_1}$ by (R2)[37].

Definition 11. Let $\boldsymbol{J_2}$ be a system defined by the axiom schema:

(Ax.1) $\Box\alpha$, whenever α is a tautology of ***CPC***,

and the following rules of inference:

(R1) $\alpha, \Box(\alpha \to \beta) / \beta$,
(R2) $\Diamond\alpha / \alpha$,
(R3) $\Box(\alpha \to \beta) / \Box(\Box\alpha \to \beta)$,

whenever β is fully modalised[38],

(R4) $\Box(\alpha \to \beta) / \Box(\alpha \to \Box\beta)$,

whenever α is fully modalised.

Remark 11. Each schema of tautology of ***CPC*** and the following rules can be taken as an inferential base for $\boldsymbol{S_5}$:

(MP),
(i) $\alpha \to \beta / \Box\alpha \to \beta$,
(ii) $\alpha \to \beta / \alpha \to \Box\beta$[39].

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[37] Jerzy Kotas was the first who used this method to prove *Theorem 7*. However, it is not difficult to see that the method is very similar to a proof of so-called Glivenko's theorem (For every α *For*: α is a thesis of classical logic iff α is a thesis of intuitionistic logic).

[38] N.C.A. da Costa gave a definition of such a formula: "We say that a formula α is fully modalised if every occurrence of every propositional variable in α falls within the scope of an occurrence of one of the operators $\Box$ or $\Diamond$.", [9], p. 10.

Theorem 6. $\boldsymbol{J} = \boldsymbol{J_2}$.

2.3. Axiomatization of $\boldsymbol{J_3}$

To show that Jaśkowski's system is finitely axiomatizable, Jerzy Kotas, in 1974, presented an interesting axiomatization of $\boldsymbol{J_3}$[40].

Definition 12. Let $\boldsymbol{J_3}$ be a system defined by the axiom schemata:

(Ax.1) $\Box (\alpha \to (\quad \alpha \to \beta))$,
(Ax.2) $\Box ((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$,
(Ax.3) $\Box ((\quad \alpha \to \alpha) \to \alpha)$,
(Ax.4) $\Box (\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$,
(Ax.5) $\Box (\Box\alpha \to \alpha)$,
(Ax.6) $\Box (\quad \Box\alpha \to \Box \quad \Box\alpha)$,

and the following rules of inference:

(R1) $\Box\alpha, \Box(\alpha \to \beta) / \Box\beta$,
(R2) $\Box\alpha / \Box \Box\alpha$,
(R3) $\Box\alpha / \alpha$,
(R4) $\Box \quad \alpha / \alpha$[41].

For convenience, we enrich $\boldsymbol{J_3}$ adding to it the possibility connective ($\Diamond$) and the rule: (R4)' $\Diamond\alpha / \alpha$.

Remark 12. Kurt Gödel proved that if we add

(i) $\Box\alpha \to \alpha$,
(ii) $\Box (\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$,
(iii) $\Box\alpha \to \Box \; \Box \alpha$

to a set of axiom of ***CPC*** and take (MP) and (RG) as the rules of inference we obtain the system $\boldsymbol{S_5}$ of Lewis[42].

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39 We impose the same restrictions on (ii) and (iii) as on (R3) and (R4), respectively. A.N. Prior is the author of this axiomatization. See [9], p. 11.

40 See [26], p. 195.

41 The author of [26] originally used propositional variables and substitution rule (Sub).

Theorem 7. $\boldsymbol{J} = \boldsymbol{J_3}$.

2.4. Axiomatization of $\boldsymbol{J_1^*}$.

By (*) added to the symbol $\boldsymbol{J_i}$[43] we understand any discursive system enriched with the quantifiers. The effect of it is to obtain a new system such as $\boldsymbol{J_1^*}$ (or $\boldsymbol{S_5^*}$) which is simply thought to be an extension of $\boldsymbol{J_1}$ ($\boldsymbol{S_5}$).

Definition 13. $\boldsymbol{J_1^*}$ and $\boldsymbol{S_5^*}$ have the same language.

Definition 14. $\boldsymbol{J^*} = \{\alpha \quad For_{S5^*} : \Diamond\alpha \quad \boldsymbol{S_5^*}\}$[44].

If $\boldsymbol{J_1^*}$ is an extension of $\boldsymbol{J_1}$, then $\boldsymbol{J_1} \quad \boldsymbol{J_1^*}$. The difference between the axiomatizations lies in the presence of the additional axiom schema:

(Ax6) $\Box(\quad _x\alpha(x) \rightarrow \alpha(t))$ where t is a variable free for x in $\alpha(x)$;

and the rule:

(R3) $\Box(\beta \rightarrow \alpha(x)) / \Box(\beta \rightarrow \quad _x\alpha(x))$ where the variable x does not occur free in β;

in the system $\boldsymbol{J_1^*}$.

Remark 13. If we enrich $\boldsymbol{S_5}$ with the axiom schema:

(v) $\quad _x\alpha(x) \rightarrow \alpha(t)$ where t is a variable free for x in $\alpha(x)$,

and the rule:

(R3) $(\beta \rightarrow \alpha(x)) / (\beta \rightarrow \quad _x\alpha(x))$ where the variable x does not occur free in β;

we obtain a complete axiomatization of $\boldsymbol{S_5^*}$.

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[42] K. Gödel did not consider axiom schemata but propositional formulae and additionally (Sub). See [26], p. 196.

[43] Where i=1,2,3,... and the sign i is an index of one of the discursive systems.

[44] See [9], p. 11 and compare with *Definition 8*.

Theorem 8. $J^* = J_1^*$.

3. Axiomatics for D_2

The paper of 1977 in which da Costa and Dubikajtis introduced the axiomatics for D_2 deserves our special attention. An *innovation* was to use the discursive connectives occurring directly in a set of axiom schemata.

Theorem 9. We may axiomatize the system D_2 by means of the following axiom schemata and rules:

(Ax.1) $\alpha \to_d (\beta \to_d \alpha)$,
(Ax.2) $(\alpha \to_d (\beta \to_d \gamma)) \to_d ((\alpha \to_d \beta) \to_d (\alpha \to_d \gamma))$,
(Ax.3) $((\alpha \to_d \beta) \to_d \alpha) \to_d \alpha$,
(Ax.4) $(\alpha \;\; _d \beta) \to_d \alpha$,
(Ax.5) $(\alpha \;\; _d \beta) \to_d \beta$,
(Ax.6) $\alpha \to_d (\beta \to_d (\alpha \;\; _d \beta))$,
(Ax.7) $\alpha \to_d (\alpha \quad \beta)$,
(Ax.8) $\beta \to_d (\alpha \quad \beta)$,
(Ax.9) $(\alpha \to_d \gamma) \to_d ((\beta \to_d \gamma) \to_d ((\alpha \quad \beta) \to_d \gamma)))$,
(Ax.10) $\alpha \to_d \quad \alpha$,
(Ax.11) $\quad \alpha \to_d \alpha$,
(Ax.12) $(\alpha \quad \alpha) \to_d \beta$,
(Ax.13) $(\alpha \quad \beta) \to_d \quad (\beta \quad \alpha)$,
(Ax.14) $(\alpha \quad \beta) \to_d (\quad \alpha \;\; _d \quad \beta)$,
(Ax.15) $(\quad \alpha \;\; \beta) \to_d \quad (\alpha \quad \beta)$,
(Ax.16) $(\quad (\alpha \quad \beta) \to_d \gamma) \to_d ((\quad \alpha \to_d \beta) \quad \gamma))$,
(Ax.17) $((\alpha \quad \beta) \quad \gamma) \to_d \quad (\alpha \quad (\beta \quad \gamma))$,
(Ax.18) $((\alpha \to_d \beta) \quad \gamma) \to_d (\alpha \;\; _d \;\; (\beta \quad \gamma))$,
(Ax.19) $((\alpha \;\; _d \beta) \quad \gamma) \to_d (\alpha \to_d \quad (\beta \quad \gamma))$,
(Ax.20) $(\quad (\alpha \quad \beta) \quad \gamma) \to_d (\quad (\quad \alpha \quad \gamma) \quad (\quad \beta \quad \gamma))$,
(Ax.21) $(\quad (\alpha \to_d \beta) \quad \gamma) \to_d (\alpha \to_d \quad (\quad \beta \quad \gamma))$,
(Ax.22) $(\quad (\alpha \;\; _d \beta) \quad \gamma) \to_d (\alpha \;\; _d \;\; (\quad \beta \quad \gamma))$,
(MP)* $\alpha, \alpha \to_d \beta / \beta$,
(RD1) $\alpha \to \beta = (\quad \alpha \quad \beta)$,
(RD2) $O\alpha = \quad (\alpha \quad \alpha)$,
(RD3) $\Box \alpha = (\quad \alpha \to_d O\alpha)$,
(RD4) $\Diamond \alpha = \quad \Box \quad \alpha$,

(Rᴅ5) $\alpha \quad \beta = (\quad \alpha \quad \beta)$,
(Rᴅ6) $\alpha \leftrightarrow \beta = ((\alpha \to \beta) \quad (\beta \to \alpha))$.

Remark 14.(Ax.1)-(Ax.22) are not independent[45].

Remark 15. The axiom schemata:

(i) $\Box((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$,
(ii) $\Box(\alpha \to (\quad \alpha \to \beta))$,
(iii) $\Box((\quad \alpha \to \alpha) \to \alpha)$,
(iv) $\Box((\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \quad \beta) \to \gamma)))$,
(v) $\Box(\alpha \to (\alpha \quad \beta))$,
(vi) $\Box(\beta \to (\alpha \quad \beta))$,
(vii) $\Box(\Box(\alpha \to \beta) \to \Box(\Box\alpha \to \Box\beta))$,
(viii) $\Box(\Box\alpha \to \alpha)$,
(ix) $\Box(\alpha \to \Box\Diamond\alpha)$

and the rules of inference:

(R1) $\alpha, \Box(\alpha \to \beta) / \beta$,
(R2) $\Diamond\alpha / \alpha$

constitute a complete axiomatization of ***J***[46].

All the axiom schemata and rules of inference of ***J*** (*Remark 15*) are provable on the ground of ***D2*** (*Theorem 9*).

3.1. Deductive structure of ***D2***[47].

[28] raised the issue of a possibility of constructing ***D2*** by the method of natural deduction. It took two years to solve the question and to present a deductive structure of ***D2*** in the sense of Jaśkowski[48] and Gentzen[49]. Before introducing the structure of ***D2*** let us recall a definition:

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[45] For details, see [1].
[46] Compare with ***J1***.
[47] To call the system, the authors used the symbols ***SD2***.
[48] See Jaśkowski, S., "On the rule of suppositions in formal logic", *Studia Logica*, 1, 1934, pp. 1-32.
[49] See Gentzen, G., "Untersuchungen über das logische Schliessen", *Mathematische Zeitschrift*, 39, 1934, pp. 176-210, 405-431.

Definition 15. $\neg\alpha = (\alpha\ {}_d(\beta\ \ \beta))$ where α, β *For* and $\neg$ is a sign of strong negation.

Theorem 10. The following rules of inference constitute a complete axiomatization of ***D*2**:

(R1) = MP*,
(R2) $\alpha\ {}_d\beta / \alpha$, (R2)′ $\alpha\ {}_d\beta / \beta$,
(R3) $\alpha, \beta / \alpha\ {}_d\beta$,
(R4) $\alpha\ \ \beta, \neg\beta / \alpha$,
(R5) $(\alpha\ \ \alpha) / \beta$
(R6) $\alpha\ {}_d\neg\beta\ /\ (\alpha\ \ \beta)$,
(R7) $(\alpha\ \ \beta)\ /\ \alpha\ {}_d\ \beta$,
(R8) $\neg(\alpha\ \ \beta) / \neg\alpha\ {}_d\neg\beta$,
(R9) $(\alpha\ {}_d\beta)\ /\ \alpha\ \ \beta$,
(R10) $\neg(\alpha \rightarrow_d \beta) / \alpha\ {}_d\neg\beta$,
(R11) $(\alpha \rightarrow_d \beta) / \alpha\ {}_d\ \beta$
(R12) $((\alpha\ {}_d\beta)\ \ \gamma), \alpha\ /\ (\beta\ \ \gamma)$,
(R13) $(\ (\alpha \rightarrow_d \beta)\ \ \gamma), \alpha\ /\ (\ \beta\ \ \gamma)$,
(R14) $(\ (\alpha\ \ \beta)\ \ \gamma)\ /\ (\ \alpha\ \ \gamma)\ \ (\ \beta\ \ \gamma)$
(R15) $(\ \alpha\ {}_d\beta)\ \ \gamma)\ /\ (\ \beta\ \ \gamma)$,
(R16) $(\alpha\ \ \beta)\ /\ (\beta\ \ \alpha)$
(R17) $((\alpha\ \ \beta)\ \ \gamma)\ /\ (\alpha\ \ (\beta\ \ \gamma))$
(R18) $((\alpha \rightarrow_d \beta)\ \ \gamma)\ /\ (\alpha \rightarrow_d (\beta\ \ \gamma))$,
(R19) $(\ \ \alpha\ \ \beta)\ /\ (\alpha\ \ \beta)$

Now, we establish rules for the construction of proofs. We may consider each formula (F) as follows:

(F) $\alpha_1 \rightarrow (\alpha_2 \rightarrow (\alpha_3 \rightarrow \ldots \rightarrow (\alpha_{n-1} \rightarrow \alpha_n)\ldots)), (n \geq 1)$.

A direct proof of (F) is formed in the following way:

(i) we write $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ in the first $n-1$ lines as suppositions of the proof;
(ii) we may add as new lines of the proof:
- formulae already proved;
- formulae obtained from the preceding lines of the proof by means of (R1)-(R19);

(iii) the proof is finished when we obtain α_n.

An indirect proof of (F) is similarly formed. The rule (i) is the same. We write $\neg\ \alpha_n$ in the n^{th} line as a supposition of the indirect proof. We may add formulae as new lines of the indirect proof just as in direct proof. The proof is finished when we obtain two lines containing any formulae of the form: α and $\neg\alpha$, α and $\neg\alpha$ or α and $\neg\ \ \alpha$

Remark 16. All the theses (axioms) of $\boldsymbol{D_2}$ are provable in (R1)-(R19)[50].

4. M^n-counterparts and *J*

Having come thus far in our consideration, it is proper to inquire whether the system $\boldsymbol{S_5}$ of Lewis is the only foundation of ***J***. In this part of the paper we wish to present the known solutions of the problem. It will be helpful to adopt a demonstration given in [29] or [34] and use it in an effort to make our trace clear. The nub of the problem is to point out and investigate the smallest and the largest normal modal systems whose M^n-counterparts coincide with the discursive system.

Definition 16. $\Diamond^n (X) = \{\alpha : \Diamond^n\alpha \quad X\}$ where X stands for a normal modal system and $n = 1, 2, 3...$[51]

As we have seen, Jaśkowski defined his calculus by an interpretation in $\boldsymbol{S_5}$ of Lewis, which has set a pattern for the further investigations. In 1975, Furmanowski presented an axiomatization of $\Diamond\boldsymbol{S_4}$ taking the rules of $\boldsymbol{J_3}$ (*Definition 12*) without (Ax6) and proved that for any normal modal system S such that $\boldsymbol{S_4} \quad S \quad \boldsymbol{S_5}$, the equality $\Diamond\boldsymbol{S_4} = \Diamond\boldsymbol{S_5}$ is valid[52]. The next results were closely related to this fact.

To bring out our exposition let us present the following axiom schemata and then put them in order:

(Ax.1) $\Box^n (\alpha \to (\ \ \alpha \to \beta))$,
(Ax.2) $\Box^n ((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$,
(Ax.3) $\Box^n ((\ \ \alpha \to \alpha) \to \alpha)$,
(Ax.4) $\Box^n (\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$,

[50] To analyse the proofs of the theses, see [28], pp. 432-435.
[51] Comp.[34], pp. 64-65.
[52] See [17].

(Ax.5) $\Box^n (\Box\alpha \to \alpha)$,
(Ax.6) $\Box^n (\Box^n \alpha \to \Box\Box^n \alpha)$,
(Ax.7) $\Box^n (\Diamond^n \Box^n \alpha \to \Box^n \alpha)$,
(Ax.8) $\Box^n (\Diamond^n \Box^n (\Box\alpha \to \alpha))$,
(Ax.9) $\Box^n (\Box^n \alpha \to \Diamond^n \Box\Box^n \alpha)$, $(n \geq 1)$.

Apply the same procedure to a set of the rules:

(R1) $\Box^n \alpha, \Box^n (\alpha \to \beta) / \Box^n \beta$,
(R2) $\Box^n \alpha / \Box\Box^n \alpha$,
(R3) $\Box^n \alpha / \alpha$,
(R4) $\Diamond^n \alpha / \alpha$,
(R5) $\Box^n \Diamond^n \Diamond^n \alpha / \Box^n \Diamond^n \alpha$,
(R6) $\Box^n \Diamond^n \alpha / \alpha$, $(n \geq 1)$

and moreover:

(RD1) $\alpha \quad \beta = \quad \alpha \to \beta$,
(RD2) $\alpha \quad \beta = \quad (\quad \alpha \quad \beta)$,
(RD3) $\alpha \leftrightarrow \beta = (\alpha \to \beta) \quad (\beta \to \alpha)$,
(RD4) $a \quad b = \Box(\alpha \to \beta)$,
(RD5) $a \Leftrightarrow b = \Box(\alpha \leftrightarrow \beta)$,
(RD6) $\Diamond \alpha = \quad \Box \quad \alpha$,
(**RD**) = {(RD1) - (RD6)}.

Next, let us modify *Definition 8*:

Definition 8'. $\boldsymbol{J} = \{\alpha : \Diamond^n\alpha \quad \boldsymbol{S5_n}\}$, $(n \geq 1)$.

From the above we obtain:

Theorem 11. We can axiomatize the systems $\Diamond^n\boldsymbol{S5_n}$, $\Diamond^n\boldsymbol{S4_n}$, $\Diamond^n\boldsymbol{T^*_n}$, $\Diamond^n\boldsymbol{K^*_n}$ $(n\geq1)$ by means of the following axiom schemata and rules:

(i) $\Diamond^n\boldsymbol{S5_n}$ by (Ax.1) - (Ax.7), (R1) - (R4) and (**RD**);
(ii) $\Diamond^n\boldsymbol{S4_n}$ by (Ax.1) - (Ax.6), (R1) - (R4) and (**RD**);
(iii) $\Diamond^n\boldsymbol{T^*_n}$ by (Ax.1) - (Ax.5), (R1) - (R5) and (**RD**);

(iv) $\Diamond^n$***K*n*** by (Ax.1) - (Ax.4), (Ax.8), (Ax.9), (R1) - (R6) and **(RD)**[53].

Remark 17. If we replace in $\Diamond^n$***T*n*** the rule (R5) with (Ax.9) or

(Ax.9)' $\Box^n$ ($\Box^n$ $\Diamond\Diamond^n$ α → $\Diamond^n$ α), ($n \geq 1$)

we obtain the systems equivalent to it.

Remark 18. $\Diamond$***K*** = .

Theorem 12. (i) $\Diamond^n$***S5n*** = $\Diamond^n$***S4n***,
(ii) $\Diamond^n$***S4n*** = $\Diamond^n$***T*n***,
(iii) $\Diamond^n$***T*n*** = $\Diamond^n$***K*n***, ($n \geq 1$)[54].

5. Algebraic approach

Da Costa and Dubikajtis were the first who used an algebraic approach to the discursive logic. The approach was further developed by Kotas. The authors made an essential contribution to the development in this field.

5.1. First algebraic semantics for the system of ***D2****.*

The authors of [12] presenting the first semantic characterisation confined themselves to giving the infinite matrix for the system of ***D2***[55]. Let **B** = <*B*, ∩, , −, 0, 1> be a Boolean algebra with meet (∩), join ( ), complement (−) and the zero (0) and unit (1) elements. Let *v* be a valuation of sentential variables of the language of ***D2*** (in the sense of [12]) into the set *B*. The function (mapping) will be extended to the function *v*' on the set of all formulae of ***D2*** into the set *B* in the following way:

(i) *v*'(α) = − (*v*'(α));
(ii) *v*'(α β) = *v*'(α) *v*'(β);
(iii) *v*'(α β) = *v*'(α) ∩ *v*'(β);
(iv) *v*'(α → β) = − (*v*'(α)) *v*'(β);
(v) *v*'(α ↔ β) = (*v*'(α) ∩ *v*'(β)) (− *v*'(α) ∩ − *v*'(β));

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53 Comp. with [29], pp. 59-62.

54 For details, see [5], [6], [7], [14], [29], [34].

55 It was further proved that the discursive logic did not have a finite characteristic matrix. See [9], pp. 8-9 and [12].

(vi) $v'(\ \alpha) = 1$ if $v'(\alpha) = 1$, otherwise $v'(\alpha) = 0$;
(vii) $v'(\ \alpha) = 0$ if $v'(\alpha) = 0$, otherwise $v'(\alpha) = 1$;
(viii) $v'(\alpha \rightarrow_d \beta) = 1$ if $v'(\alpha) = 0$, otherwise $v'(\alpha \rightarrow_d \beta) = v'(\beta)$;
(ix) $v'(\alpha \leftrightarrow_d \beta) = 1$ if $v'(\alpha) = v'(\beta) = 0$, otherwise $v'(\alpha \leftrightarrow_d \beta) = 0$ if $v'(\alpha) = 0 \neq v'(\beta)$, $v'(\alpha \leftrightarrow_d \beta) = v'(\beta)$ if $v'(\alpha) \neq 0$;
(x) $v'(\alpha \ \ _d \beta) = 0$ if $v'(\alpha) = 0$, otherwise $v'(\alpha \ \ _d \beta) = v'(\alpha)$[56].

Any function v': *For* B fulfilling (i)-(x) is a valuation of the language into the Boolean algebra **B**.

Theorem 13. For every formula α : the formula α is a thesis of ***D*₂** iff for an arbitrary Boolean algebra **B** and valuation v' of formulae in **B** the inequality $v'(\alpha) \neq 0$ is true[57].

*5.2. Henle algebra and the system **J***

By Henle algebra **H** = < *A*, −, , I> we shall understand a Boolean algebra <*A*, −, > with a unary (interior) operation I over *A*:

$$Ia = \begin{cases} 1 \text{ if } a = 1, \\ 0 \text{ if } a \neq 1. \end{cases}$$

The reader should realise that a Henle algebra is intended as semantics for the modal system ***S*₅** and if we consider the definition of ***J*** , we shall have the equality: ***J*** = ◇***S*₅**. It also suffices to notice that the possibility connective has the same properties as the closure operation *C* over *A*:

$$Ca = \begin{cases} 1 \text{ if } a \neq 0, \\ 0 \text{ if } a = 0. \end{cases}$$

Theorem 14. For every modal formula α the following conditions are equivalent:

(i) α is a thesis of ***J*** ,
(ii) for every finite Henle algebra **H** and valuation v of formulae in **H**, $v(\alpha) \neq 0$,

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[56] [12], pp. 554-555. See also [27], p. 154.
[57] For a proof of *Theorem 13*, see [12], p. 555.

(iii) for every valuation v of formulae in the Henle algebra **H** = $< A, -, \ , I >$, where $< A, -, \ >$ is the Boolean algebra of all finite and cofinite sets of natural numbers, $v(\alpha) \neq 0$[58].

5.3. Discursive algebra

What we intended to do is integrally linked with the earlier exposition. We wish to present the infinite matrix for $\boldsymbol{D_2}$. For the sake of a concise explanation, it is convenient to consider the language: $\boldsymbol{L_{D2}} = < For_{D2}, \ , \ , _d, \rightarrow_d, \leftrightarrow_d >$ and *the discursive algebra* **A** similar to $\boldsymbol{L_{D2}}$:

Definition 17. An algebra $\mathbf{A} = < A, -, \ , _d, \rightarrow_d, \leftrightarrow_d, 0, 1>$ is called to be *discursive* when the following conditions hold:

(i) the structure $\mathbf{A} = <A, -, \ 0, 1>$ is a Boolean algebra, for every elements $a, b \ A$:

(ii) $$a \rightarrow_d b = \begin{cases} 1 \text{ if } a = 0, \\ b \text{ if } a \neq 0; \end{cases}$$

(iii) $$a \ _d b = \begin{cases} 0 \text{ if } b = 0, \\ a \text{ if } b \neq 0; \end{cases}$$

(iv) $$a \leftrightarrow_d b = \begin{cases} 1 \text{ if } a = b = 0, \\ 0 \text{ if } a = 0 \neq b, \\ b \text{ if } a \neq 0. \end{cases}$$

Now let us consider the languages of $\boldsymbol{D_2}$ and $\boldsymbol{S_5}$:

(i) $\boldsymbol{L_{D2}} = < For_{D2}, \ , \ , _d, \rightarrow_d >$,
(ii) $\boldsymbol{L_{S5}} = < For_{S5}, \ , \ , \ , \rightarrow \Diamond, \Box >$.

We determined the interpretation $f: For_{D2} \ \ For_{S5}$ (*Definition 6*).We presently proceed to indicate *a class of all the discursive matrices* of the form: $\mathbf{K_D} = \{< \boldsymbol{A}, A - \{0\} > : \boldsymbol{A}$ is a discursive algebra$\}$.

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[58] [9], p. 8, see also [27], p. 155.

Definition 7. For every $\alpha \quad \boldsymbol{L_{D2}}$: $\alpha \quad \boldsymbol{D_2}$ iff $\Diamond f(\alpha) \quad \boldsymbol{S_5}$.

Theorem 15. $\alpha \quad \boldsymbol{L_{D2}}$: $\alpha \quad \boldsymbol{D_2}$ iff $m \quad \mathbf{K_D}$, $h \quad$ Hom ($\boldsymbol{L_{D2}}$, **A**), $h(\alpha)\neq 0$.

We prove the theorem by means of Henle algebra.

5.4. Congruence difficulties

In an attempt to point out an algebraic approach to $\boldsymbol{D_2}$ we have to abandon the standard approach to the problem. A relation we define by means of discursive equivalence turned out not to be congruence of the algebra $\boldsymbol{L_{D2}}$. The standard methods failed. In spite of the difficulties, Kotas showed that with the help of the remaining connectives we could define a relation, which is congruence of this algebra.

5.4.1. Relation of equivalence

In order to obtain it let us consider:

Definition 18. For any $\alpha, \beta \quad For$:

$$\alpha \approx \beta \text{ iff } \alpha \rightarrow_d \beta ,\ \beta \rightarrow_d \alpha \quad \boldsymbol{D_2},$$

Remark 19. Relation $\approx$ is an equivalence and, moreover, $\alpha_1 \approx \alpha_2$ and $\beta_1 \approx \beta_2$, imply:

(i) $\alpha_1 \quad \beta_1 \approx \alpha_2 \quad \beta_2$,
(ii) $\alpha_1 \quad {}_d\beta_1 \approx \alpha_2 \quad {}_d\beta_2$,
(iii) $\alpha_1 \rightarrow_d \beta_1 \approx \alpha_2 \rightarrow_d \beta_2$,
(iv) $\alpha_1 \leftrightarrow_d \beta_1 \approx \alpha_2 \leftrightarrow_d \beta_2$,

however, it is not true that:

(v) $\alpha_1 \approx \alpha_2$.

5.4.2. Implicative lattice

Having rejected (v), we have to limit ourselves to the set of negationless formulae. Let *For'* be the set of formulae without negation. Notice that relation $\approx$ is a congruence of the algebra $\boldsymbol{L'} = \langle For', \ , \ {}_d, \rightarrow_d \rangle$. Let $\boldsymbol{L^*} = \boldsymbol{L'}/\approx$ and $\ , \cap, \rightarrow$ be operations in $\boldsymbol{L^*}$ introduced by connectives $\ , \ {}_d, \rightarrow_d$, respectively.

Theorem 16. The algebra $< \boldsymbol{L}^*, \ , \cap, \rightarrow >$ is an implicative lattice.

To prove *Theorem 16* it is necessary to show that:

(i) $< \boldsymbol{L}^*, \ , \cap >$ is a lattice;
(ii) for arbitrary elements a, b, x $\boldsymbol{L}^*$:

(L1) $(a \cap (a \rightarrow b)) \quad b = b$,
(L2) if $(a \cap x) \quad b = b$, then $x \quad (a \rightarrow b) = a \rightarrow b$,
(L3) $a \quad (a \rightarrow b) = a \rightarrow a$,
(L4) $((a \rightarrow b) \rightarrow a) \quad a = a$.

It is worth mentioning that an implicative lattice (relatively pseudo-complemented lattice in the terminology of Rasiowa)[59] holds due to (L1) and (L2), but when we add (L3) and (L4) to them we obtain a very special implicative lattice: *Boolean* (or *classical*).

5.4.3. Implication **(↑d)** *of* D2

Kotas in [24] gave a definition of the logical system with implication[60] and indicated that the discursive implication ($\rightarrow_d$) did not satisfy the fourth property. The question is to find such a connective which is definable by means of the remaining ones (, , $_d$) and satisfies all the properties. As a result, he defined an operation ($\uparrow_d$):

Definition 19. $\alpha \uparrow_d \beta = \quad ((p \quad p) \quad_d \quad (\quad \alpha \quad \beta))$

Definition 20. $\alpha \rightarrow_d \beta = \quad ((p \quad p) \quad_d \alpha) \quad \beta$.

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[59] See Rasiowa H., "An algebraic approach to non-classical logics", *PWN*, 1974.

[60] A logical system **L** will be called a logical system with implication, if among connectives of the system there exists or is definable at least 2-ary connective denoted by $\rightarrow$ having the following properties: (i) $(\alpha \rightarrow \alpha)$ **L**; (ii) if α **L** and $(\alpha \rightarrow \beta)$ **L**, then β **L**; (iii) if $(\alpha \rightarrow \beta)$ **L** and $(\beta \rightarrow \gamma)$ **L**, then $(\alpha \rightarrow \gamma)$ **L**; (iv) for any k_i-ary logical operation O of **L**: if $(\alpha_j \rightarrow \beta_j)$ **L** and $(\beta_j \rightarrow \alpha_j)$ **L**, j = 1,2,…,k_i, then $(O(\alpha_1,\ldots,\alpha_{k_i}) \rightarrow O(\beta_1,\ldots,\beta_{k_i}))$ **L** and $(O(\beta_1,\ldots,\beta_{k_i}) \rightarrow O(\alpha_1,\ldots,\alpha_{k_i}))$ **L**. See [24], p. 102.

Remark 20. The operation $\uparrow_d$ is an implication in $\boldsymbol{D_2}$, that is, it has the properties (i)-(iv)[61].

Definition 21. For arbitrary formulae α and β:

$$\alpha \equiv \beta \text{ iff } \alpha\uparrow_d\beta,\ \beta\uparrow_d\alpha \quad \boldsymbol{D_2};$$

Remark 20 and *Definition 21* imply that the relation $\equiv$ is congruence in the algebra of formulae of $\boldsymbol{D_2}$.

Remark 21. The Lindenbaum algebra of $\boldsymbol{D_2}$ contains minimal and maximal elements, but not each of its theses represents its maximal element.

Now, we may construct a Lindenbaum's algebra of $\boldsymbol{D_2}$. Let $<\boldsymbol{D_2}/\equiv,\ ,\ ,\cap>$ be a quotient algebra, where $\ ,\ ,\cap$ are the operations in $\boldsymbol{D_2}/\equiv$ introduced by the connectives $\ ,$ and $_d$, respectively.

Theorem 17. The algebra $<\boldsymbol{D_2}/\equiv,\ ,\ ,\cap>$ has the following properties:

(i) $<\boldsymbol{D_2}/\equiv,\ ,\ >$ is a Boolean algebra;

(ii) $<\boldsymbol{D_2}/\equiv,\ ,\cap>$ is a skew lattice, that is: $<\boldsymbol{D_2}/\equiv,\ >$ is a semilattice with the zero ($0 = [\ (p \quad p)]$) and unit ($1 = [p \quad p)]$) elements and the following conditions hold:

(L1) $a \cap (b \cap c) = (a \cap b) \cap c$,
(L2) $(a \cap b) \quad c = a \cap (b \quad c)$,
(L3) $a \cap b \cap a = a \cap b$,
(L4) $a \cap (b \quad c) = (a \cap b) \quad (a \cap c)$,
(L5) $(a \quad b) \cap c = (a \cap c) \quad (b \cap c)$,

(iii) for any $a, b, c \quad \boldsymbol{D_2}/\equiv$:

(L6) $a \cap b = \ (\ a \quad (1 \cap b))$,
(L7) $1 \cap 0 = 0$,
(L8) $a \quad (1 \cap a)) = 1 \cap a$.

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[61] See [27], p. 157.

Conclusions

The discussion presented in this paper is far from being exhaustive. Jaśkowski formulated only the propositional part of his logic. It was not a difficult task to extend it to a first (or higher) order logic. We have hardly mentioned this achievement. Since the choice by Jaśkowski is related to modal systems we have also ignored many-valued approaches. The reader can find the solution of the problem in [15] and [30]. In the end, let us mention that $\boldsymbol{D_2}$ does not have a finite characteristic matrix (neither does ***J***). These results were proved in [25] and [27].

As we have seen, the standard approach to the problem of algebraization of $\boldsymbol{D_2}$ fails. Despite this difficulty we can construct Lindenbaum's algebra for $\boldsymbol{D_2}$ with the help of , and d. We defined the relation (≡) that was congruence of the algebra.

We may now consider the question of how Jaśkowski's logic is related to the so-called paraconsistent one. What should be emphasised is the fact that Jaśkowski was the first who touched upon this problem. He was in quest of a calculus having the following property: *when it is applied to the contradictory systems will not always entail their overcompleteness*. In fact, he built the first formal system of propositional paraconsistent logic[62].

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[62] Comp. [3], p. 33 and [11], p. 112.

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