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NATURAL DEDUCTION FOR PARACONSISTENT LOGIC*

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Abstract
In this paper, by using the method of natural deduction, *via* the method of subordinate proofs, we develop a hierarchy of natural deduction logical systems NDC_n containing just deduction rules (or deduction schemata) with no axiom schema. We prove that these systems NDC_n, $1 \leq n \leq \omega$, are logically equivalent to the systems of Da Costa's hierarchy of paraconsistent logics C_n, $1 \leq n \leq \omega$. Some of the deduction rules used to introduce these systems are new and do not correspond to Da Costa's axioms rewritten, permitting the definition of a new paraconsistent semantics, such that soundness and completeness of the systems NDC_n, $1 \leq n < \omega$, may be directly obtained. Other natural deduction systems logically equivalent to Da Costa's systems C_n, $1 \leq n \leq \omega$, are also introduced.

1. Introduction

A deductive theory T is said to be inconsistent if it has as theorems a formula and its negation; otherwise, T is said to be consistent. A deductive theory T is said to be trivial if every formula of its language is a theorem; otherwise, T is said to be non-trivial.

If a theory T has as its underlying logic the classical logic, the deduction of a contradiction leads to its trivialization. Therefore, in theories based on classical logic, to deduce a contradiction is equivalent to trivialize them.

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A logic is paraconsistent if it can be used as the underlying logic to inconsistent but non-trivial theories, which we call paraconsistent theories.

D'Ottaviano [8] points that in paraconsistent logic the role of the Principle of Non-contradiction is, in a certain sense, restricted. Although in those logics the Principle of Non-contradiction is not necessarily invalid, from a formula and its negation it is not possible, in general, to deduce any other formula.

Da Costa [5] introduced his hierarchy of first-order logics for the study of inconsistent yet non-trivial theories: the hierarchy of propositional calculi $\mathbf{C}_n$, $1\leq n\leq\omega$, the hierarchy of predicate calculi $\mathbf{C}_n$, $1\leq n\leq\omega$, the hierarchy of predicate calculi with equality $\mathbf{C}_n^{=}$, $1\leq n\leq\omega$, and the hierarchy of calculi of descriptions $\mathbf{D}_n$, $1\leq n\leq\omega$.

Da Costa, his disciples and collaborators have used the Hilbert-Fregean style axiomatic method in the study of Da Costa's hierarchy $\mathbf{C}_n$, $1\leq n\leq\omega$. The first work in which this Hilbert-Fregean style was not used in the construction of paraconsistent systems is due to Raggio [17].

Raggio [17] introduces a hierarchy of *sequent* calculi $\mathbf{CG}_n$, $1\leq n\leq\omega$, trying to solve the problem of the decidability of the systems $\mathbf{C}_n$, $1\leq n\leq\omega$. Raggio proves the equivalence between the *sequent* calculi $\mathbf{CG}_n$, $1\leq n\leq\omega$, and the calculi $\mathbf{C}_n$, $1\leq n\leq\omega$, but $\mathbf{CG}_n$ could not be proved decidable. Continuing, he constructs a new hierarchy of *sequent* calculi $\mathbf{WG}_n$, $1\leq n\leq\omega$, that are decidable, but in spite of having similar properties to the ones of the systems $\mathbf{C}_n$, $1\leq n\leq\omega$, these two hierarchies of systems are not equivalent.

In Alves [1], systems of natural deduction were introduced for the hierarchy $\mathbf{C}_n$, $1\leq n\leq\omega$, using only deduction rules, but unfortunately Alves did not study those systems.

As both axiomatizations for the two equivalent systems $\mathbf{C}_\omega$* and $\mathbf{CG}_\omega$*, introduced by Da Costa and Raggio respectively, were not suitable for a proof-theoretic analysis, Raggio [18] presents the system $\mathbf{NC}_\omega$*, logically equivalent to Da Costa's system $\mathbf{C}_\omega$*, using Gentzen's natural deduction. This axiomatic system of quantificational paraconsistent logic without equality has the peculiarity of having the Law of Excluded Middle as its sole axiom. Pereira and Moura [16], an unpublished work, presents a natural deduction system with no axiom, the system $\mathbf{NNC}_\omega$, that improves the propositional part of $\mathbf{NC}_\omega$* introduced in Raggio [18].

Castro [3] applies the method of natural deduction introduced by Jaskowski [11] and Gentzen [10], through the method of subordinate proofs of Fitch [9], to the hierarchy of Da Costa's propositional paraconsistent logics $\mathbf{C}_n$, $1 \leq n \leq \omega$.

In this paper, based in Castro [3], we introduce the hierarchy of natural deduction systems $\mathbf{NDC}_n$, $1 \leq n \leq \omega$, and show that $\mathbf{NDC}_1$ is logically equivalent to $\mathbf{C}_1$; and we sketch the necessary procedures to demonstrate this equivalence between $\mathbf{NDC}_n$ and $\mathbf{C}_n$, $2 \leq n \leq \omega$. In spite of the main syntactical and semantical results, like for instance consistency, soundness and completeness being natural consequences of the logical equivalence between the hierarchies $\mathbf{C}_n$ and $\mathbf{NDC}_n$, $1 \leq n \leq \omega$, we may prove the soundness and completeness of our systems directly from the definition of a new paraconsistent semantics, which we introduce in this paper.

In the next section we present Da Costa's propositional paraconsistent systems $\mathbf{C}_n$, $1 \leq n \leq \omega$, and some important results about them.

In the third section, we recall the method of natural deduction by subordinate proofs, and introduce a new natural deduction formulation for the paraconsistent logic $\mathbf{C}_1$, the system $\mathbf{NDC}_1$.

In the fourth section, we prove the logical equivalence between $\mathbf{C}_1$ and $\mathbf{NDC}_1$.

In the fifth and sixth sections, we present a new natural deduction formulation for the paraconsistent logic $\mathbf{C}_n$, $1 \leq n \leq \omega$, the systems $\mathbf{NDC}_n$, $1 \leq n \leq \omega$.

In the last section, we discuss some results of the previous sections and introduce a new paraconsistent semantics relatively to Alves [1]. We also formulate two new hierarchies of paraconsistent systems such that in every one of these hierarchies there is a system logically equivalent to the corresponding system $\mathbf{C}_n$, $1 \leq n \leq \omega$.

We observe that some of the rules that we use to introduce the systems $\mathbf{NDC}_n$, $1 \leq n \leq \omega$, are new, not simply corresponding to Da Costa's axioms rewritten, as it was done by Alves [1].

2. Da Costa's propositional paraconsistent logics $\mathbf{C}_n$

The language **L** of Da Costa's paraconsistent systems $\mathbf{C}_n$, $1 \leq n \leq \omega$, has as primitive symbols propositional variables, the connectives $\neg$, , &, , and parentheses.

The notions of formula and theorem, as well as the general conventions and notations, are the standard ones, as in Kleene [12].

Da Costa’s paraconsistent logics were formulated satisfying the conditions:

> "I - In $\mathbf{C}_1$ it should not be valid, in general, the principle of non contradiction "
>
> “II - From two contradictory propositions it should not usually be possible to deduce any proposition”.

The system $\mathbf{C}_1$ is the first of the hierarchy of systems of propositional paraconsistent logics presented by Da Costa [5]. The following definitions are added to the language **L**:

$A^0 =_{df} \neg(A\&\neg A)$
$A^n =_{df} A^{0\ldots0}$ (‘0’ n times)[1]
$A^{(1)} =_{df} A^0$
$A^{(n)} =_{df} A^1 \,\&\, A^2 \,\&\ldots\&\, A^n$,
with $n \geq 1$, where A^1 is A^0, A^2 is A^{00}, A^3 is A^{000}, …, A^n is $A^{0\ldots0}$
$(A\equiv B) =_{df} (A \quad B)\&(B \quad A)$
$\quad A =_{df} \neg A\&A^0$.

According to these definitions, A^0 is read as “A is a well-behaved formula” or “A is regular”; A^n is read as “A is a n-times reiterated regular formula”; $A^{(n)}$ is read as “A is a composed regular formula of degree n”; the symbol ≡ corresponds to the usual equivalence; and the connective   is called “strong negation”.

We observe that the strong negation has all the properties of the classical negation, since we can prove that $(A \quad B) \quad ((A \quad\; B) \quad\; A)$, for every A and B formulas of $\mathbf{C}_1$.

The schemata of axioms and the deduction rules of $\mathbf{C}_1$ are the following:

AXIOM 1: $A \quad (B \quad A)$
AXIOM 2: $(A \quad B) \quad (((A \quad (B \quad C)) \quad (A \quad C))$
AXIOM 3: $A\&B \quad A$
AXIOM 4: $A\&B \quad B$
AXIOM 5: $A \quad (B \quad A\&B)$

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[1] For instance: A^{00} is $\neg(A^0\&\neg(A^0))$; A^{000} is $\neg(A^{00}\&\neg(A^{00}))$; and so on.

AXIOM 6: A   A   B
AXIOM 7: B   A   B
AXIOM 8: (A   C)   ((B   C)   (A   B   C))
AXIOM 9: ¬¬ A   A
AXIOM 10: A   ¬A
AXIOM 11: B^0 ((A   B)   ((A   ¬B)   ¬A))
AXIOM 12: A^0&B^0 $(A\&B)^0$
AXIOM 13: A^0&B^0 $(A\ \ B)^0$
AXIOM 14: A^0&B^0 $(A\ \ B)^0$
RULES: Rule of *Modus Ponens* (MP)
Substitution Rule (SR).

Axiom 11 corresponds to Da Costa's insight of paraconsistent logic. This axiom is just a particular case of the usual *reductio ad absurdum*. It asserts that we can apply the *reductio ad absurdum* in Da Costa's paraconsistent logic only when the antecedent B is a "well-behaved" formula.

The last three axioms may be interpreted as the conditions for propagation of "well-behavior".

In order to obtain the systems $\mathbf{C}_n$, $1 \leq n < \omega$, we replace Axioms 11 to 14:

AXIOM 11^n: $B^{(n)}$ ((A   B)   ((A   ¬B)   ¬A))
AXIOM 12^n: $A^{(n)}$&$B^{(n)}$ $(A\&B)^{(n)}$
AXIOM 13^n: $A^{(n)}$&$B^{(n)}$ $(A\ \ B)^{(n)}$
AXIOM 14^n: $A^{(n)}$&$B^{(n)}$ $(A\ \ B)^{(n)}$

In every $\mathbf{C}_n$, $1 \leq n < \omega$, the strong negation   is defined by:

$$_nA =_{df} \neg A\&A^{(n)},$$

where n corresponds to each $\mathbf{C}_n$.

We observe that in every $\mathbf{C}_n$ the strong negation has all the properties of classical negation, since we can prove (A   B)   ((A $_n$B) $_n$A)) in $\mathbf{C}_n$, $1 \leq n \leq \omega$.

Finally, the system $\mathbf{C}_\omega$ is defined by:

AXIOM 1 to AXIOM 10 of $\mathbf{C}_1$.

We observe that *classical propositional logic* is considered as the system $\mathbf{C}_0$ of this hierarchy. This logic is, in fact, given by $\mathbf{C}_\omega$ plus *reductio ad absurdum*, that is:

AXIOM 11º: (A B) ((A ¬B) ¬A)).

Now, only in order to better characterize these systems, we state some of their results. Some of the proofs of these results are in the papers mentioned in D'Ottaviano [8].

Theorem 2.1. *The following schemata are not provable in* $\mathbf{C}_n$, $1 \leq n \leq \omega$:

¬(A&¬A); (A&¬A) B; A ¬¬A; ¬A≡ $_n$A;
A (¬A B); ((A B)&¬A) B; $(A ¬A)^n$; $(A A)^n$;
((A B)&¬B) ¬A; ¬(A B)≡¬A&¬B; ¬(A&B)≡¬A ¬B;
¬(A B)≡A&¬B; ¬(¬A&A) ((A B) ((A ¬B) ¬A));
¬(¬A&A) ¬(A&¬A).

Theorem 2.2. *In* $\mathbf{C}_n$, $1 \leq n < \omega$ *we have*:

$\vdash_{C_n}$ A $_n$A; $\vdash_{C_n}$ A≡ $_n$ $_n$A; $\vdash_{C_n}$ ¬(A&B) ¬A ¬B;
$\vdash_{C_n}$ ¬ $_n$A A; $\vdash_{C_n}$ $_n$¬A A; $\vdash_{C_n}$ $_n$(A& $_n$A);
$\vdash_{C_n}$ A& $_n$A B; $\vdash_{C_n}$ $(A\&\neg A)^n$; $\vdash_{C_n}$ $(A)^n$;
$\vdash_{C_n}$ $(A)^n$ $(\neg A)^n$; $\vdash_{C_n}$ A ($_n$A B); $\vdash_{C_n}$¬(A&¬A) ¬(¬A&A).

Theorem 2.3. *All the rules and valid schemata of the classical positive propositional calculus are valid in* $\mathbf{C}_n$, $1 \leq n < \omega$.

Theorem 2.4. *The following schemata are not provable in* $\mathbf{C}_\omega$:

((A B) A) A; (A B) (B A); ¬(A&B) ¬A ¬B.

Theorem 2.5. *No schema of type* ¬A *is a theorem in* $\mathbf{C}_\omega$.

Theorem 2.6. *If* A *is a theorem of the intuitionistic positive propositional calculus, then* A *is a theorem of* $\mathbf{C}_\omega$.

Theorem 2.7. (*Deduction Theorem*) *If* Γ *is a set of formulas, we have that* Γ, A $\vdash_{C_n}$ B *if, and only if,* Γ $\vdash_{C_n}$ A B, $1 \leq n \leq \omega$.

The following theorem concerns regular formulas.

Theorem 2.8. *If* Γ *is a set of formulas and* $A_1, A_2, \ldots, A_m$ *are the atomic components of the formulas of* $\Gamma \quad \{A\}$, *then a necessary and sufficient condition for* $\Gamma \vdash_{C_0} A$ *is that* $\Gamma, A_1^{(n)}, A_2^{(n)}, \ldots, A_m^{(n)} \vdash_{C_n} A$, *for* $1 \leq n < \omega$.

Theorem 2.9. (**Arruda**) *Every system in the hierarchy* $\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_n, \ldots, \mathbf{C}_\omega$ *is strictly stronger than those which follow it*.

Definition 2.10. Let   be the set of all formulas of **L**. A set of formulas Γ is said to be *trivial* if the set of consequences of Γ is ; Γ is said to be *inconsistent* (relatively to the basic negation $\neg$) if there is at least one formula A such A and $\neg$A are both consequences of Γ.

Theorem 2.11. *Every* $\mathbf{C}_n$, $1 \leq n \leq \omega$, *is consistent and non-trivial*.

Theorem 2.12. (**Arruda**) *The systems* $\mathbf{C}_n$, $1 \leq n \leq \omega$, *are not decidable by finite matrices*.

We observe that the *Replacement Theorem*[2], although valid in $\mathbf{C}_0$, is not valid in general in $\mathbf{C}_n$, $1 \leq n \leq \omega$.

A very recent study on the systems $\mathbf{C}_n$, $1 \leq n \leq \omega$, from a new semantical approach, appears in Marcos [15].

3. The method of natural deduction applied to the paraconsistent logic $\mathbf{C}_1$

In this section, we introduce the system of natural deduction $\mathbf{NDC}_1$, using the method of natural deduction *a la* Fitch [9]. The language of $\mathbf{NDC}_1$ is the language of $\mathbf{C}_1$.

We adopt thirteen deduction rules, which allow us to deduce exactly the theorems of the axiomatic system $\mathbf{C}_1$. The system $\mathbf{NDC}_1$ is the first of a hierarchy of systems of natural deduction for Da Costa's paraconsistent propositional logics $\mathbf{C}_n$, $1 \leq n \leq \omega$.

The system $\mathbf{NDC}_1$ is introduced through the following rules:

[2] This special theorem of substitution states: if A and B are equivalent, they may be substituted for each other at any point in an expression C.

Rule of transport

Repetition (R)

:	:		
n	A		
:	:		
k	A		(R, n)

Reiteration (R)

:	:		
n	A		
:		| :	
k		| A	(Reit, n)

Introduction rules

Implication Introduction (I-)

:	:		
k		A	supposition
:		| :	
s		| B	
s+1	A B		k-s, I -

Disjunction Introduction (I-)

:	:		
k	A	(or B)	
:	:		
s	A B	(or A B)	k, I -

Conjunction Introduction (I-&)

:	:		
k	A	(or B)	
:	:		
s	B	(or A)	
:	:		
u	A&B		k, s, I - &

Restricted Rule of Negation Introduction [or restricted *Reductio ad Absurdum*] (I - ¬ (rest))

: :
p A^0
: :
k | B supposition
: | :
r | A (or ¬A)
: | :
t | ¬A (or A)
v ¬B p, k-t, I - ¬(rest)

Distributive Rule of Negation into Conjunction (DNC)

: :
p ¬(A&B)
: :
s ¬A ¬B p, DNC

Distributive Rule of Negation into Disjunction (DND(rest))

: :
p A^0 (or B^0)
: :
q B^0 (or A^0)
: :
r ¬(A B)
: :
s ¬A&¬B p, q, r, DND(rest)

Distributive Rule of Negation into Implication (DNI(rest))

: :
p A^0 (or B^0)
: :
q B^0 (or A^0)
: :
r ¬(A B)
: :
s A&¬B p, q, r, DNI(rest)

Non-Constructive Dilemma (NDC)

:	:			
p		|	A	supposition
:		|	:	
r		|	B	
s		|	¬ A	supposition
:		|	:	
t		|	B	
t+1	B			p-r, s-t, NCD

Elimination rules

Implication Elimination (E -)

:	:		
p	A B	(or A)	
:	:		
q	A	(or A B)	
:	:		
r	B		p, q, E -

Disjunction Elimination (E -)

:	:			
p	A B			
:	:			
q		|	A	supposition
:		|	:	
r		|	C	
s		|	B	supposition
:		|	:	
t		|	C	
t+1	C			p, q-r, s-t, E -

Conjunction Elimination (E - &)

:	:	
p	A&B	
:	:	
q	A	(or B) p, E - &

Double Negation Elimination (E - $\neg\neg$)

:	:	
p	$\neg\neg$ A	
:	:	
q	A	p, E - $\neg\neg$

A *formal proof* in **NDC**$_1$ is a finite sequence of items (formulas) where each one of them is either a premise (or hypothesis), an axiom of the system, or is logically derived from previous ones in the sequence by application of only one deduction rule. A formal proof that possesses one or more premises is said to be a *hypothetical proof*, and a formal proof that has no premises is called a *categorical proof*. A *subordinate proof* (of a given proof) is a proof that begins with an additional premise (or *supposition*). All subordinate proofs are subordinated to a principal proof and all of them must be eliminated in order to return to the main ones.

If a proof and all its subordinate proofs (if any) use only deduction rules, it will be said to be an *introduction-elimination proof* (or an *intelim proof*).

If there is a proof of B_n in **NDC**$_1$ from the premises A_1, A_2, … A_n, this is denoted by A_1, A_2, … $A_n \vdash_{\mathbf{NDC}1} B_n$.

If B_n is the final item of a formal proof, then this formula is said to be *provable* or to be a *conclusion*.

If there is a proof of B_n in **NDC**$_1$ from the empty set of premises, this formula is said to be a *theorem*, what is denoted by $\vdash_{\mathbf{NDC}1} B_n$.

Graphically, we represent a proof in **NDC**$_1$ by a vertical sequence of items (formulas or subproofs), and we develop a subproof in a parallel vertical sequence of items to the immediate right of the principal sequence of items.

Let **NDC**$_0$ be the classical system of natural deduction as in Gentzen [10]. The difference between **NDC**$_1$ and **NDC**$_0$ is not only in the amount of adopted rules, but also in the restrictions imposed to certain rules of deduction. In **NDC**$_1$, for instance, the application of the rule of *reductio ad absurdum* is conditioned to the previous presence of a certain regular formula in the demonstration, while in **NDC**$_0$ this is not necessary.

4. The logical equivalence between the system $\mathbf{C_1}$ and the system $\mathbf{NDC_1}$

In order to prove the syntactical equivalence between Da Costa's paraconsistent system $\mathbf{C}_1$ and our natural deduction system $\mathbf{NDC}_1$, we shall first prove that every theorem of $\mathbf{C}_1$ is provable in $\mathbf{NDC}_1$; second, we shall prove that every deduction rule of the system $\mathbf{NDC}_1$ is deducible in the axiomatic system $\mathbf{C}_1$.

Theorem 4.1. *Every theorem of the system* $\mathbf{C}_1$ *is a theorem of the system of natural deduction* $\mathbf{NDC}_1$.

Proof

We have to prove that every axiom schema of $\mathbf{C}_1$ is a theorem in $\mathbf{NDC}_1$. In order to illustrate this we shall only present the complete proofs of Axioms 11 and 12. The proofs of Axiom 1-10 are simple and we shall not make them. The proofs of Axioms 13 and 14 are similar to the proof of Axiom 12.

1 Axiom 11: $B^0 \quad ((A \quad B) \quad ((A \quad \neg B) \quad \neg A))$

1	B^0	supposition
2	$A \quad B$	supposition
3	$A \quad \neg B$	supposition
4	A	supposition
5	$A \quad B$	2, Reit
6	B	4, 5, E -
7	$A \quad \neg B$	3, Reit
8	$\neg B$	4, 7, E -
9	$\neg A$	1, 4- 8, I - ¬(rest)
10	$(A \quad \neg B) \quad \neg A$	3-9, I -
11	$(A \quad B) \quad ((A \quad \neg B) \quad \neg A)$	2-10, I -
12	$B^0 \quad ((A \quad B) \quad ((A \quad \neg B) \quad \neg A))$	1-11, I -

2 Axiom 12: A^0&B^0 $(A\&B)^0$

1	A^0&B^0	supposition
2	A^0	1, E - &
3	B^0	1, E - &
4	¬[$(A\&B)^0$]	supposition
5	¬[¬[(A&B)&¬(A&B)]]	4, def. of regular formula
6	[(A&B)&¬(A&B)]	5, E - ¬¬
7	A&B	6, E - &
8	A	7, E - &
9	B	7, E - &
10	¬(A&B)	6, E - &
11	¬A ¬B	10, DNC
12	¬A	supposition
13	¬A	12, R
14	¬B	supposition
15	A	supposition
16	B	9, Reit
17	¬B	14, Reit
18	¬A	3, 15-17, I -¬(rest)
19	¬A	11, 12-13, 14-18, E -
20	¬¬ [$(A\&B)^0$]	2, 4-19, I - ¬(rest)
21	$(A\&B)^0$	20, E - ¬¬
22	A^0&B^0 $(A\&B)^0$	1-21, I -

The rule of *Modus Ponens* of $\mathbf{C}_1$ corresponds to the Implication Elimination Rule of $\mathbf{NDC}_1$.

For every application of the Substitution Rule in a theorem of $\mathbf{C}_1$, there is a corresponding proof in $\mathbf{NDC}_1$.

We observe that as usually in axiomatic systems and in natural deduction systems, $\mathbf{C}_1$ and $\mathbf{NDC}_1$ have the following *Assertion Property* (AP):

If $\vdash_{\mathbf{S}}$ A then $\Gamma \vdash_{\mathbf{S}}$ A, for every set Γ of formulas of the language of **S**, **S** being either $\mathbf{C}_1$ or $\mathbf{NDC}_1$.

Furthermore:

If $\Gamma \vdash_{\mathbf{C}_1}$ A, then $\Gamma \vdash_{\mathbf{NDC}_1}$ A.

Theorem 4.2. *Every deduction rule of the system* **NDC**$_1$ *is provable in the axiomatic system* **C**$_1$**.**

Proof

We shall indicate the proofs of the rules DNC, DND(rest) and DNI(rest), because they need a special sequence of steps. The proofs of the other rules are found in the literature.

1) The proof that A $\vdash_{C_1}$ A is immediate.

2) Distributive Rule of Negation into Conjunction (DNC)

The proof of DNC is an immediate consequence of Theorem 2.2 ($\vdash_{C_n}$ ¬(A&B)   (¬A   ¬B)) and Theorem 2.7.

3) Distributive Rule of Negation into Disjunction (DND(rest))

Proof

We have to prove that:

A^0& B^0, ¬(A B) $\vdash_{C_1}$ ¬A&¬B.

By (1), AP, Axiom 1, SR, MP, Axiom 7, Axiom 11, Axiom 6, Axiom 5 and Deduction Theorem:

(a) $\vdash_{C_1}$ $(A\ \ B)^0$ (¬(A B) (B (¬A&¬B))).

By (1), AP, Axiom 1, SR, MP, Axiom 6, Axiom 11, Axiom 5 and Deduction Theorem:

(b) $\vdash_{C_1}$ $(A\ \ B)^0$ (¬(A B) (¬B (¬A&¬B))).

By (1), AP, $(A\ \ B)^0$ $\vdash_{C_1}$ ¬(A B) (B ¬A&¬B), $(A\ \ B)^0$ $\vdash_{C_1}$ ¬(A B) (¬B ¬A&¬B), MP, Axiom 10, SR, Axiom 8 and Deduction Theorem:

(c) $\vdash_{C_1}$ $(A\ \ B)^0$ (¬(A B) (¬A&¬B)).

By Axiom 13, SR, $(A\ \ B)^0$ $\vdash_{C_1}$ ¬(A B) (¬A&¬B), Deduction Theorem, transitivity of implication and MP:

(d) $\vdash_{C_1}$ A^0&B^0 (¬(A B) (¬A&¬B)).

By (1), AP, (c), Axiom 13, SR, transitivity of implication and MP:

(e) A^0& B^0, ¬(A B) $\vdash_{C_1}$ ¬A&¬B.

4) Distributive Rule of Negation into Implication (DNI(rest))

Proof

We have to prove that:

$A^0 \& \ B^0, \neg(A\ \ \ B) \vdash_{\mathbf{C}_1} (A \& \neg B)$.

The demonstration is obtained through the following sequence of steps.

By (1), AP, $A \& B \vdash_{\mathbf{C}_1} A$, SR, Axiom 5, MP, distributivity of & into $, \vdash_{\mathbf{C}_1} A^0 \& A \& \neg A\ \ \ B$, Axiom 4, Axiom 8 and Deduction Theorem:

(a) $A^0 \& B^0 \vdash_{\mathbf{C}_1} \neg A\ \ B\ \ \ (A\ \ \ B)$.

By (1), AP, Axiom 11, SR, Axiom 1, MP, Axiom 14 and Deduction Theorem:

(b) $A^0 \& B^0 \vdash_{\mathbf{C}_1} (\neg A\ \ B\ \ \ (A\ \ \ B))\ \ \ (\neg(A\ \ \ B)\ \ \ \neg(\neg A\ \ \ B))$.

By (a), (b) and MP:

(c) $A^0 \& B^0 \vdash_{\mathbf{C}_1} \neg(A\ \ \ B)\ \ \ \neg(\neg A\ \ \ B)$.

By (1), Axiom 3, SR, AP, MP, Axiom 4, Axiom 9, Axiom 5 and Deduction Theorem:

(d) $A^0 \vdash_{\mathbf{C}_1} (\neg A \& \neg\neg A)\ \ \ (A \& \neg A)$.

By (1), Axiom 11, SR, AP, MP, (d), Axiom 1, definition of regular formula and Deduction Theorem:

(e) $\vdash_{\mathbf{C}_1} A^0\ \ \ (\neg A)^0$.

By (1), Axiom 3, SR, AP, MP, (e), Axiom 4 and Axiom 5:

(f) $A^0 \& B^0 \vdash_{\mathbf{C}_1} (\neg A)^0 \& B^0$.

By (f), 3(d), SR, AP and MP:

(g) $A^0 \& B^0, \neg(\neg A\ \ \ B) \vdash_{\mathbf{C}_1} (\neg\neg A \& \neg B)$.

By (g), Axiom 3, SR, Axiom 9, transitivity of implication, MP, AP, Axiom 4, Axiom 5 and Deduction Theorem:

(h) $A^0 \& B^0 \vdash_{\mathbf{C}_1} \neg(\neg A\ \ \ B)\ \ \ (A \& \neg B)$.

By (c), (h), transitivity of implication, SR, AP and MP:

(i) $A^0 \& B^0 \vdash_{\mathbf{C}_1} \neg(A\ \ \ B)\ \ \ (A \& \neg B)$.

By (1), AP, (i) and MP:

(j) $A^0 \& B^0, \neg(A\ \ \ B) \vdash_{\mathbf{C}_1} (A \& \neg B)$.

So, we have proved that, if $\Gamma \vdash_{\mathbf{DNC}_1} A$, then $\Gamma \vdash_{\mathbf{C}_1} A$. Hence, by Theorem 4.1 and Theorem 4.2,

$$\Gamma \vdash_{\mathbf{DNC}_1} A \text{ if, and only if } \Gamma \vdash_{\mathbf{C}_1} A.$$

5. The method of natural deduction applied to the paraconsistent logics $\mathbf{C_n}$

In this section, we introduce the natural deduction systems $\mathbf{NDC}_n$, for $1<n<\omega$.

For every logical system $\mathbf{NDC}_n$, $1<n<\omega$, we also adopt thirteen deduction rules and these allow us to deduce all the provable formulas of the correspondent axiomatic systems $\mathbf{C}_n$.

Each natural deduction system $\mathbf{NDC}_n$, for $0\leq n<\omega$, is deductively stronger than $\mathbf{NDC}_{n+1}$, and this property is transmitted to every one of the strong negations “ $_n$”.

In every logical system $\mathbf{NDC}_n$, $1<n<\omega$, specific restrictions are imposed on some of the deduction rules. As for instance, in every $\mathbf{NDC}_n$, $1<n<\omega$, the application of the *reductio ad absurdum* is conditioned to the previous appearing in the proof of an adequate composed regular formula of degree n.

The rules of deduction of $\mathbf{NDC}_n$**,** $1<n<\omega$, have the same formulations given in $\mathbf{NDC}_1$, excepting for the following three cases:

Restricted Principle of Negation Introduction [or *Reductio ad Absurdum* restricted] (I - $\neg_n$ (rest))

:	:		
p	$A^{(n)}$		
:	:		
k		| B	supposition
:		| :	
r		| A	(or $\neg A$)
:		| :	
t		| $\neg A$	(or A)
v	$\neg B$		p, k- t, I - $\neg_n$(rest)

Distributive Rule of Negation into Disjunction (DND$_n$(rest))

:	:	
p	$A^{(n)}$	(or $B^{(n)}$)
:	:	
q	$B^{(n)}$	(or $A^{(n)}$)
:	:	
r	$\neg(A\ \ B)$	
:	:	

s $\neg A \& \neg B$ p, q, r, DND_n(rest)

Distributive Rule of Negation into Implication (DNI_n(rest))

: :
p $A^{(n)}$ (or $B^{(n)}$)
: :
q $B^{(n)}$ (or $A^{(n)}$)
: :
r $\neg(A \quad B)$
: :
s $A \& \neg B$ p, q, r, DNI_n(rest)

The logical equivalence between every system $\mathbf{NDC}_n$ and the corresponding $\mathbf{C}_n$, $1<n<\omega$, is obtained following step by step the procedures developed for the case $\mathbf{NDC}_1$ and $\mathbf{C}_1$.

6. The method of natural deduction applied to the paraconsistent logic $\mathbf{C}_\omega$

In this section, we introduce a natural deduction system, the system $\mathbf{NDC}_\omega$, equivalent to the paraconsistent logic $\mathbf{C}_\omega$. The rules of $\mathbf{NDC}_\omega$ are the same as presented to $\mathbf{NDC}_1$, without the Rules E - ¬(rest), DNC, DND(rest) and DNI(rest).

The proof of the logical equivalence between the systems $\mathbf{NDC}_\omega$ and $\mathbf{C}_\omega$ is immediate, from the previous sections.

We observe that the logical equivalence between the propositional part of the system $\mathbf{NC}_\omega$*, introduced in Raggio [18], and our system $\mathbf{NDC}_\omega$ is immediate.

A natural deduction system introduced by Alves [1] and the system $\mathbf{NNC}_\omega$ presented by Pereira and Moura [16] have the same deduction rules as our system $\mathbf{NDC}_\omega$. Nevertheless, in our work, we obtain the system $\mathbf{NDC}_\omega$ by a natural construction, from the hierarchy $\mathbf{DNC}_\mathbf{n}$, $1\leq n<\omega$.

7. Final remarks

Although the proof of the logical equivalence between Da Costa's axiomatic systems $\mathbf{C}_\mathbf{n}$, $1\leq n<\omega$, and our natural deduction systems guarantees the soundness and completeness of the systems $\mathbf{NDC}_\mathbf{n}$, $1\leq n<\omega$, we developed these syntactical and semantical results for the natural deduction systems $\mathbf{NDC_n}$, $1 \leq n < \omega$. Our goal in improving these results is to obtain an autonomous development to the systems $\mathbf{NDC_n}$, $1 \leq n < \omega$.

Alves [1] introduces the concept of paraconsistent valuation and quasi-matrices, and proves soundness, completeness and decidability of the system $\mathbf{C}_1$.

Loparic [13], based on Alves's paper proves soundness, completeness and decidability of the system $\mathbf{C}_\omega$.

Loparic and Alves [14], based on Alves [1] and Da Costa and Alves [7], modify the conditions of Alves's definitions of valuation and prove soundness, completeness and decidability of the systems $\mathbf{C}_n$, $1 \leq n < \omega$.

The definition introduced by Loparic and Alves is the following:

Definition. If   is the set of formulas of $\mathbf{C}_n$, $1 \leq n < \omega$, a *valuation* for $\mathbf{C}_n$ is a function ν: $\rightarrow \{0, 1\}$ such that:

1. If $\nu(A) = 0$, then $\nu(\neg A) = 1$;
2. If $\nu(\neg\neg A) = 1$, then $\nu(A) = 0$;
3. $\nu(A\&B) = 1$ if, and only if, $\nu(A) = 1$ and $\nu(B) = 1$;
4. $\nu(A\ \ B) = 1$ if, and only if, either $\nu(A) = 1$ or $\nu(B) = 1$;
5. $\nu(A\ \ B) = 1$ if, and only if, either $\nu(A) = 0$ or $\nu(B) = 1$;
6. If $\nu(A^{n-1}) = \nu(\neg A^{n-1})$, then $\nu(A^n) = 0$;
7. If $\nu(A) = \nu(\neg A)$, then $\nu(\neg A^1) = 1$;
8. If $\nu(A) \neq \nu(\neg A)$, $\nu(B) \neq \nu(\neg B)$, then $\nu((A\#B)) \neq \nu(\neg(A\#B))$,

where # is &,   or .

We can introduce a new semantics directly connected to $\mathbf{NDC}_n$, $1 \leq n < \omega$, in which a valuation for $\mathbf{NDC}_n$, $1 \leq n < \omega$, is a function ν: $\rightarrow \{0, 1\}$ such that the following conditions (7), (8) and (9) replace the condition 8 above:

7. If $\nu(\neg(A\&B)) = 1$, then $\nu(\neg A) = 1$ or $\nu(\neg B) = 1$;
8. If $\nu((A)^{(n)}) = \nu((B)^{(n)}) = \nu(\neg(A\ \ B)) = 1$, then $\nu(\neg A\&\neg B) = 1$;
9. If $\nu((A)^{(n)}) = \nu((B)^{(n)}) = \nu(\neg(A\ \ B)) = 1$, then $\nu(A\&\neg B) = 1$.

By using our new definition of paraconsistent valuation, the properties of maximal consistent sets can be extended to maximal non-trivial sets and we can directly prove the soundness and completeness of the systems $\mathbf{NDC_n}$, $1 \leq n < \omega$.

Another observation concerns the deductive efficiency of $\mathbf{NDC}_n$, $1 \leq n < \omega$, relative to the formulation presented in Alves [1]. Alves introduces natural deduction systems for the $\mathbf{C}_n$, $1 \leq n \leq \omega$, through the following rules:

$$_i \ \frac{\begin{matrix}[A]\\ B\end{matrix}}{A \quad B} \qquad _e \ \frac{A \quad A \quad B}{B} \qquad \&_i \ \frac{A \qquad B}{A\&B}$$

$$\&_e \ \frac{A\&B}{A} \quad \frac{A\&B}{B} \qquad _i \ \frac{A}{A \quad B} \quad \frac{B}{A \quad B}$$

$$e \ \frac{A \quad B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

$$o_i' \ \frac{A^{(n)} \quad B^{(n)}}{(A \quad B)^{(n)}} \qquad o_i'' \ \frac{A^{(n)} \quad B^{(n)}}{(A\&B)^{(n)}} \qquad o_i''' \ \frac{A^{(n)} \quad B^{(n)}}{(A \quad B)^{(n)}}$$

$$\neg_1 \ \frac{\begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[\neg A]\\ C\end{matrix}}{C} \qquad \neg_2 \ \frac{\neg\neg A}{A} \qquad \neg_3 \ \frac{A^{(n)} \quad A \quad \neg A}{B}$$

In these systems, instead of our Rule I - $\neg_n$(rest), we find the Rule $\neg_3$. Alves's formulation is sustained by the following result stated by Da Costa:

> "We could see, without great difficulties that in C_n the postulate '$B^{(n)}$ ((A B) ((A ¬B) ¬A))' can be substituted by schema $B^{(n)}$&B&¬B K".

In fact, the rules o_i', o_i'', o_i''' above constitute a transliteration of Da Costa's Axioms 12, 13 and 14, respectively.

We observe that the Rules DNC, DND_n(rest), DNI_n(rest) of our systems $\mathbf{NDC}_n$ are new, not rewritten from Da Costa's axioms like Alves's rules o', o'' and o'''. The Rule I - $\neg_n$(rest) emphasizes that in $\mathbf{NDC}_n$,(and $\mathbf{C}_n$), $1 \leq n < \omega$, the role of the Principle of Non-contradiction is, in a certain sense, restricted. We think that these rules and I - $\neg_n$(rest) are better applicable to actual derivations in mathematical proofs.

For example, it is very easy to derivate $A^{(n)} \& B^{(n)} \quad (A \& B)^{(n)}$ in $\mathbf{NDC}_n$, $1 \leq n < \omega$, but, it is difficult to derivate $\neg(A \& B) \quad (\neg A \quad \neg B)$ in Alves's systems.

A final consideration results from the study of structure of proofs in $\mathbf{NDC}_n$. It allows us to formulate two new natural deduction systems logically equivalent to $\mathbf{NDC}_n$, $1 \leq n < \omega$.

A first system equivalent to $\mathbf{NDC}_n$ is obtained by substitution of Rule I - $\neg_n$(rest) by the following rule:

Restricted Principle of Negation Elimination [E - $\neg_n$(rest)]

:	:		
p		$\neg C$	supposition
:		:	
q		$B^{(n)}$	
:		:	
r		B (or $\neg B$)	
:		:	
s		$\neg B$ (or B)	
t	C		p-s, E - $\neg_n$(rest)

Another system which is logically equivalent to $\mathbf{NDC}_n$ is obtained by substitution of Rule I - $\neg_n$(rest) by:

Restricted Principle of Negation Introduction [I_2 - $\neg_n$(rest)]

:	:		
p		C	supposition
:		:	
q		$B^{(n)}$	
:		:	
r		B (or $\neg B$)	
:		:	
s		$\neg B$ (or B)	
t	$\neg C$		p-s, I_2 - $\neg_n$(rest).

If we adopt I_2 - $\neg_n$(rest) as our primitive deduction rule in **NDC**$_n$, we can prove that the Non-Constructive Dilemma(NDC) is a derived rule.

We observe that, while in our original systems **NDC**$_n$, $1 \leq n < \omega$, the Rule I - $\neg_n$(rest) emphasizes the non-contradiction, in these two systems the Rules E - $\neg_n$(rest) and I_2 - $\neg_n$(rest) emphasize the weak negation of the systems.

Carnielli and Lima-Marques [4] and Buchsbaum & Pequeno [2] introduce tableaux type systems equivalent to the systems $\mathbf{C}_1^=$ and $\mathbf{C}_1^*$, respectively, and prove the decidability of these systems.

In a future paper we shall present and analyze a new hierarchy of tableaux systems **TNDC**$_n$, $1 \leq n < \omega$, equivalent to the hierarchy **NDC**$_n$, $1 \leq n < \omega$, comparing them to Carnielli's tableaux and Buchsbaum's tableaux, and prove the decidability of these systems. We shall prove the decidability of these systems **TNDC**$_n$, $1 \leq n < \omega$.

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REFERENCES

[1] ALVES, E.H., *Lógica e inconsistência: um estudo dos cálculos* C_n , $1 \leq n \leq \omega$, São Paulo, Universidade de São Paulo, 1976 (Master Dissertation).

[2] BUCHSBAUM, A., PEQUENO, T., "A reasoning method for a paraconsistent logic", *Studia Logica*, 1993, vol. 52, p. 281-289.

[3] CASTRO, M. A. de, *O método de dedução natural aplicado às lógicas proposicionais paraconsistentes* C_n. Campinas: Universida-de Estadual de Campinas. 1998 (Master Dissertation).

[4] CARNIELLI, W.A., LIMA-MARQUES, M., "Reasoning under inconsistent knowledge", *Journal of Applied Non-Classical Logics*, 1992, vol. 2, n.1, p. 49-79.

[5] DA COSTA, N.C.A., Sistemas formais inconsistentes. Paraná: Universidade Federal do Paraná. 1963 (Thesis).

[6] DA COSTA, N.C.A., "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic*, 1974, vol. 15, p. 497-510.

[7] DA COSTA, N.C.A, ALVES, E.H., "A semantical analysis of the cal-

culi C_n", *Notre Dame Journal of Formal Logic*, 1977, vol. 18, p. 621-630.

[8] D'OTTAVIANO, I.M.L., "On the development of paraconsistent logic and Da Costa's work", *The Journal of Non-Classical Logic,* 1990, vol. 7, no. 1/2, p. 89-152.

[9] FITCH, F.B., *Symbolic logic: an introduction*, Ronald Press, New York, 1952.

[10] GENTZEN, G., *Investigations into logical deduction* (*Untersuchungen über das logische Schliessen*), *in* SZABO, ME., ed, The Colleted Papers of Gerhard Gentzen, North-Holland Publishing, Amsterdam, 1969, p. 68-131.

[11] JASKOWSKI, S., "On the rules of suppositions in formal logic", *in* MCCALL, S., ed, *Polish Logic 1920-1939*, The Clarendon Press, Oxford, 1967, p. 232-258.

[12] KLEENE, S.C., *Introduction to metamathematics*, North Holland, Amsterdam, 1952.

[13] LOPARIC, A., "Une étude sémantique de quelques calculs propositionnels", *C.R. Acad. Sc. Paris*, 1977, vol. 284A, p. 835-838.

[14] LOPARIC, A., ALVES. E.H., "The semantics of the systems C_n of Da Costa", *in* "Brazilian Conference On Mathematical Logic", 3st. Ed. A.I. Arruda, N.C.A. Da Costa, A.M., Sette. *Proceedings*, Sociedade Brasileira de Lógica, São Paulo, 1980, p. 161-162.

[15] MARCOS, J., *Semânticas de traduções possíveis. Campinas: Universidade Estadual de Campinas*, 1999 (Master Dissertation).

[16] PEREIRA, L.C.P.D., MOURA, J.E. de A., "Some aspects of C_ω in natural deduction - version 1." [S.n.], Manuscript, to appear.

[17] RAGGIO, A.R., "Propositional sequence-calculi for inconsistent systems", *Notre Dame Journal of Formal Logic*, 1968, vol. 9, no. 4, p. 359-366.

[18] RAGGIO, A.R., "A proof-theoretic analysis of Da Costa's C_ω*", *in* "Brazilian Conference On Mathematical Logic", 1st Ed. A. I. Arruda, N.C.A. Da Costa, R. Chuaqui, *Proceedings*, Marcel Dekker, New York, 1978, p. 233-240.