Projections in free and Boolean probability as models of multi-valued logic of Łukasiewicz and Tarski and connections with Tsallis entropy S_q

Marek Bożejko

In the talk we will present the following topics:

1. Free products of groups with state (positive definite function) as models of free and Boolean probability.

2. We show that if P, Q are orthogonal projections, free independent in a probabilistic system $(\mathcal{A}, \mathrm{tr})$, where tr is a faithfull trace on a von Neumann algebra $\mathcal{A}$, then either $P \vee Q = I$, or $P \wedge Q = 0$, here $P \wedge Q$ is the projection on intersection $P(H)$ and $Q(H)$ and $P \vee Q$ is the projection onto the closed Hilbert space generated by $P(H) + Q(H)$.

   Moreover we have

   $$\mathrm{tr}(P \wedge Q) = \max(\mathrm{tr}(P) + \mathrm{tr}(Q) - 1, 0) = f_0(\mathrm{tr}(P), \mathrm{tr}(Q)),$$

   where f_0 is the function of polish logician J. Łukasiewicz discovered in 1918!, for a construction of a model of multi-valued logic. There are some connections with the results of Ben Arous and Voiculescu (Free Extreme Values, *Annals of Probability 34* (5), 2006: https://projecteuclid.org/euclid.aop/1163517232).

3. We also show the connections with Tsallis entropy

   $$S_q(p_1, ...p_N) = -\sum_i p_i \log_q(p_i)$$

   (and recent results of Voiculescu and Vargas about orthogonal projections in Boolean probability (arXiv 2017: https://arxiv.org/abs/1711.06227):

   If E, F are a Boolean independent orthogonal projections in a system $(\mathcal{A}, t)$, then

   $$t(E \wedge F) = f_2(t(E), t(F)),$$

   where

   $$f_q(x, y) = \max[x^{1-q} + y^{1-q} - 1, 0]^{1/(1-q)},$$

   where x, y are in interval $[0, 1]$, $q \in \mathbb{R}$.

Remark. For the case $q = 1$, we get classical independence in probability, i.e.

$$t(A \wedge B) = f_1(t(A), t(B)) = t(A)t(B)$$

for measurable independent sets A and B.