The logics of analytic equivalence

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Introduction

Our aim is to present some non-Fregean logics realizing, in the framework of Suszko, the following

Wójcicki's principle:

Two sentences whose logical forms in sentential language are logically equivalent and have the same sentential variables, describe the same situation [Wójcicki R., *Semantyka sytuacyjna logiki niefregowskiej* (in Polish, *Situational semantics of non-Fregean logic*) [in:] Pelc J. (ed.) Znaczenie i prawda. Rozprawy semiotyczne, WN PWN Warszawa 1994].

This principle is a particular case of

Principle of Barwise and Perry:

Two sentences whose logical forms are logically equivalent and have the same extralogical constants, describe the same situation [Barwise J., Perry J., *Semantic innocence and uncompromising situations*, Midwest Studies in Philosophy 6(1981) 387-404].

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Wittgenstein's principle:

two sentences whose logical forms are logically equivalent describe the same situation.

Analysing the ways of understanding the expression "situation" by Suszko, Wolniewicz, Barwise, Perry on one side, and the expression "proposition" in the western logical literature on the other side, we come to the following

equality:

situation = proposition, that is "to describe a situation" means "to express a proposition".

Let $\mathcal{L} = (L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv)$ be a sentential language such that $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ are standard connectives and $\equiv$ is a binary one which will be interpreted as sentential identity, in the sense that when a formula $\alpha \equiv \beta$ is valid then the same proposition is expressed by both formulas α, β. The basic non-Fregean sentential logic, well known under the name *SCI* (*sentential calculus with identity*), is a consequence relation $\vdash_{SCI}$ defined on the language $\mathcal{L}$ by means of the rule *Modus Ponens*: $\alpha, \alpha \rightarrow \beta / \beta$ and the following

axioms:

(*Cl*) a group of axioms defining the classical sentential logic,
(*Ax*1) $\alpha \equiv \alpha$,
(*Ax*2) $(\alpha \equiv \beta) \rightarrow (\neg\alpha \equiv \neg\beta)$,
$(Ax3)_f$ $((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \rightarrow ((\alpha f \gamma) \equiv (\beta f \delta))$, for $f \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$
(*Ax*4) $(\alpha \equiv \beta) \rightarrow (\alpha \leftrightarrow \beta)$.

The logic $\vdash_{SCI}$ is really basic for comparison the propositions expressed by formulas, due to its following property:

for any $\alpha, \beta \in L$, α is the same formula as β whenever $\vdash_{SCI} \alpha \equiv \beta$.

In the sequel the following valuational semantics for $\vdash_{SCI}$ will be used (originally the semantics for $\vdash_{SCI}$ provided by Bloom and Suszko is a bundle of some kind of matrices called *SCI*-models). Given any algebra $\mathcal{A} = (A, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \circ)$ similar to the language $\mathcal{L}$ and any homomorphism $h : \mathcal{L} \longrightarrow \mathcal{A}$ consider the class $Val(\mathcal{A}, h)$ of all the mappings $v : L \longrightarrow \{0, 1\}$ fulfilling for any $\alpha, \beta \in L$ the following

conditions:

$v(\neg\alpha) = 1$ iff $v(\alpha) = 0$,
$v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = v(\beta) = 1$,
$v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
$v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
$v(\alpha \leftrightarrow \beta) = 1$ iff $v(\alpha) = v(\beta)$,
$v(\alpha \equiv \beta) = 1$ iff $v(\alpha) = v(\beta)$ & $h(\alpha) = h(\beta)$.

It is clear that any such v from $Val(\mathcal{A}, h)$ is uniquely determined by its values on sentential variables and by the homomorphism h. Now, the semantics for $\vdash_{SCI}$ is the class $Val_{SCI} = \bigcup\{Val(\mathcal{A}, h) : \mathcal{A}$ - algebra similar to $\mathcal{L}$ & $h \in Hom(\mathcal{L}, \mathcal{A})\}$ in the sense that it defines the consequence relation $\models \subseteq P(L) \times L$ in the standard way (that is $X \models \alpha$ iff for each $v \in Val_{SCI}$, $v(\alpha) = 1$ whenever $\overrightarrow{v}(X) \subseteq \{1\}$) such that $\models = \vdash_{SCI}$. The proof of completeness theorem: $\models \subseteq \vdash_{SCI}$, runs along the quite standard lines by application of the Lindenbaum lemma for the logic $\vdash_{SCI}$ and the canonical valuation $v_X \in Val_{SCI}$ uniquely determined by a maximal theory X of $\vdash_{SCI}$ in the following way. The values on the sentential variables are: $v_X(p) = 1$ iff $p \in X$, for any variable p. The quotient algebra $\mathcal{L}/\!\approx$ (where a congruence relation $\approx$ is defined on L as follows: $\alpha \approx \beta$ iff $\alpha \equiv \beta \in X$) plays a role of the algebra $\mathcal{A}$ and $h : \mathcal{L} \longrightarrow \mathcal{L}/\!\approx$ is the canonical homomorphism: $h(\alpha) = [\alpha]_{\approx}$,any $\alpha \in L$.

A strengthening of *SCI* realizing the Wójcicki's principle in a natural way

Let $\models_1$ be a consequence relation defined on $\mathcal{L}$ by the class $Val_1 \subseteq Val_{SCI}$ of all the valuations $v \in Val_{SCI}$ determined by homomorphisms $h : (L, \neg, \wedge, \vee, \to, \leftrightarrow, \equiv) \longrightarrow (A, id, \cup, \cup, \cup, \cup, \cup)$, where $(A, \cup)$ is any similattice and *id* is the identity function of A, more precisely, $v \in Val_1$ iff $v \in Val(\mathcal{A}, h)$ for some semilattice $\mathcal{A} = (A, id, \cup, \cup, \cup, \cup, \cup)$ and a homomorphism $h : \mathcal{L} \longrightarrow \mathcal{A}$. First we show that the axiomatic counterpart $\vdash_1$ of $\models_1$ is a strengthening of *SCI* defined by the following

axioms:

(*Ax*5) $\alpha||\neg\alpha$,
(*Ax*6)$_f$ $(\alpha f \beta)||(\alpha \equiv \beta)$ for $f \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
(*Ax*7) $((\alpha||\beta) \wedge (\gamma||\delta)) \rightarrow ((\alpha \equiv \gamma)||(\beta \equiv \delta))$,
(*Ax*8) $(\alpha \equiv \alpha)||\alpha$,
(*Ax*9) $(\alpha \equiv \beta)||(\beta \equiv \alpha)$,
(*Ax*10) $(\alpha \equiv (\beta \equiv \gamma))||((\alpha \equiv \beta) \equiv \gamma)$,
(*Ax*11) $(\alpha \equiv \beta) \rightarrow ((\alpha \leftrightarrow \beta) \wedge (\alpha||\beta))$,
(*Ax*12) $((\alpha \leftrightarrow \beta) \wedge (\alpha||\beta)) \rightarrow (\alpha \equiv \beta)$,

where $||$ is a new binary connective of the form:
$\alpha||\beta =_{def} (\alpha \equiv \alpha) \equiv (\beta \equiv \beta)$. Notice that this connective is important due to its semantic behaviour: for any $v \in Val_1$:
$v(\alpha||\beta) = 1$ iff $h(\alpha) = h(\beta)$,
where the homomorphism h determines that v.

The proof of soundness theorem is straightforward. In proving the completeness the following theses of $\vdash_1$ are useful:
$(t1)$ $\alpha||\alpha$,
$(t2)$ $(\alpha||\beta) \to (\beta||\alpha)$,
$(t3)$ $((\alpha||\beta) \wedge (\beta||\gamma)) \to (\alpha||\gamma)$.
They are responsible (together with *Modus Ponens*) for the fact that given any maximal theory X of the logic $\vdash_1$, the relation $\approx$ defined on L by means of the clause: $\alpha \approx \beta$ iff $\alpha||\beta \in X$, is an equivalence relation. Moreover, one can show that it is in fact a congruence relation of the language. This follows from the property of the theory X : $\alpha \wedge \beta \in X$ iff $\alpha, \beta \in X$, the theses $(t2), (t3)$ and the axioms: $(Ax5), (Ax7), (Ax6)_f$. The quotient algebra $(L/\approx, \neg, \wedge, \vee, \to, \leftrightarrow, \equiv)$, due to the axioms $(Ax5), (Ax7)$, has the following operations: $\neg[\alpha] = [\alpha]$ and for any $f \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$: $[\alpha]f[\beta] = [\alpha \equiv \beta]$. Hence in fact the algebra is of the form: $(L/\approx, id, \equiv, \equiv, \equiv, \equiv, \equiv)$, where obviously $[\alpha] \equiv [\beta] = [\alpha \equiv \beta]$. In order to show that $(L/\approx, \equiv)$ is a semilattice apply $(Ax8)$ to prove the idempotence of the operation $\equiv$ as well as $(Ax9), (Ax10)$ to prove that $\equiv$ is commutative and associative respectively. Finally, consider the canonical homomorphism $h_0 : L \longrightarrow L/\approx$, that is for any $\alpha \in L$, $h_0(\alpha) = [\alpha]$. Then

we have the following

Lemma

Given any maximal theory X of $\vdash_1$ for any formulae α, β:
$(\neg)$ $\neg\alpha \in X$ iff $\alpha \notin X$,
$(\wedge)$ $\alpha \wedge \beta \in X$ iff $\alpha, \beta \in X$,
$(\vee)$ $\alpha \vee \beta \in X$ iff $\alpha \in X$ or $\beta \in X$,
$(\rightarrow)$ $\alpha \rightarrow \beta \in X$ iff $\alpha \notin X$ or $\beta \in X$,
$(\leftrightarrow)$ $\alpha \leftrightarrow \beta \in X$ iff $(\alpha \in X$ iff $\beta \in X)$,
$(\equiv)$ $\alpha \equiv \beta \in X$ iff $(\alpha \in X$ iff $\beta \in X)$ and $h_0(\alpha) = h_0(\beta)$.

Now, on the basis of the lemma, given a maximal theory X of the logic $\vdash_1$ one can show that the valuation $v_X \in Val_1$ determined by the following its values on the sentential variables: $v_X(p) = 1$ iff $p \in X$, and the homomorphism h_0, is the characteristic function of X in L, that is for any $\alpha \in L$, $v_X(\alpha) = 1$ iff $\alpha \in X$. This leads directly to completeness theorem: $\models_1 \subseteq \vdash_1$.

The following theorem in a simple way characterizes the logic $\models_1$:

Theorem 1.

For any $\alpha, \beta \in L$: $\models_1 \alpha \equiv \beta$ iff $\models_1 \alpha \leftrightarrow \beta$ and $V(\alpha) = V(\beta)$, where for each formula γ, $V(\gamma)$ is the set of all sentential variables occurring in γ.

PROOF. ($\Rightarrow$): Assume that $\models_1 \alpha \equiv \beta$ and take any $v \in Val_1$. From the assumption it follows that $v(\alpha \equiv \beta) = 1$, therefore $v(\alpha) = v(\beta)$ and consequently $v(\alpha \leftrightarrow \beta) = 1$, which results in $\models_1 \alpha \leftrightarrow \beta$. In order to show that $V(\alpha) = V(\beta)$, first observe that one can treat the symbol V as naming the homomorphism
$V : (L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv) \longrightarrow (P_{fin}(Var), id, \cup, \cup, \cup, \cup, \cup)$ (where Var is the set of all sentential variables of $\mathcal{L}$ and $P_{fin}(Var)$ is the family of all finite subsets of Var) uniquely determined by the following its values on sentential variables: $V(p) = \{p\}$, any $p \in Var$. Now, taking into account any valuation $v' \in Val_1$ determined by the homomorphism V, according to the assumption it can be obtained that $v'(\alpha \equiv \beta) = 1$, so $V(\alpha) = V(\beta)$.

(⇐): Suppose that (1) $\models_1 \alpha \leftrightarrow \beta$ and (2) $V(\alpha) = V(\beta)$. Consider any $v \in Val_1$. Obviously such a v is determined by some homomorphism $h : \mathcal{L} \longrightarrow (A, id, \cup, \cup, \cup, \cup, \cup)$, where $(A, \cup)$ is a similattice. Then one can show inductively on the length of a formula that for any $\gamma \in L$, $h(\gamma) = h(p_1) \cup \ldots \cup h(p_n)$, where $\{p_1, \ldots, p_n\} = V(\gamma)$. Hence and from (2) it follows that $h(\alpha) = h(\beta)$. Moreover, from (1): $v(\alpha) = v(\beta)$, which together gives the result: $v(\alpha \equiv \beta) = 1$, so consequently, $\models_1 \alpha \equiv \beta$.

Let us point that a part of the Wójcicki's principle - the expression "logically equivalent", is pretty ambiguous, so it can be understood in several ways. As it is obviously seen due to the theorem 1, the logic $\vdash_1$ realizes only one such a way of understanding, may be most expected or natural. For "logically equivalent" refers here to the very logic which is used to establish the identity of propositions. However, the expression could be understood more trivially: *two formulas are logically equivalent iff their equivalence is a classical tautology* - moreover, a tautology in a complete classical sense that it does not contain the connectives other than standard. Now we will deal with an appropriate logic.

A trivial strengthening of *SCI* realizing the Wójcicki's principle

Let us start from the very definition of

the second axiomatic strengthening of *SCI*:

for any $X \subseteq L$ and $\gamma \in L$: $X \vdash_2 \gamma$ iff
$X \cup \{\alpha \equiv \beta : \alpha, \beta \in L_{cl} \ \& \ \vdash_{cl} \alpha \leftrightarrow \beta \ \& \ V(\alpha) = V(\beta)\} \vdash_{SCI} \gamma$, where $\vdash_{cl}$ is the classical logic defined on the subset L_{cl} of L composed out of all the formulas in which $\equiv$ does not occur.

The most important property of the logic $\vdash_2$ is presented in the following

Theorem 2.

For any formulas $\alpha, \beta \in L_{cl}$, $\vdash_2 \alpha \equiv \beta$ iff $\vdash_{cl} \alpha \leftrightarrow \beta \ \& \ V(\alpha) = V(\beta)$.

PROOF. $(\Leftarrow)$: Obvious.

($\Rightarrow$): Assume that $\vdash_2 \alpha \equiv \beta$ for some formulas α, β in which the sentential identity $\equiv$ does not occur. So from the definition of $\vdash_2$ we have: $AX \vdash_{SCI} \alpha \equiv \beta$, where
$AX = \{\gamma \equiv \delta : \gamma, \delta \in L_{cl}, \vdash_{cl} \gamma \leftrightarrow \delta, \; V(\gamma) = V(\delta)\}$. So
(1) $AX \models \alpha \equiv \beta$,
where $\models$ is the consequence relation determined by the semantics Val_{SCI}.
In order to show that $\vdash_{cl} \alpha \leftrightarrow \beta$ suppose that it does not hold. Then there exists a classical valuation v such that
(2) $v(\alpha) \neq v(\beta)$.
So extend it to the valution $v' \in Val_{SCI}$ in the way that $v'(p) = v(p)$ for any $p \in Var$, and v' is determined by the homomorphism $h : \mathcal{L} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is any fixed 1-element algebra similar to $\mathcal{L}$. Then,
(3) for any formulas $\gamma, \delta \in L_{cl}$, $v'(\gamma \equiv \delta) = 1$ iff $v(\gamma) = v(\delta)$.
Hence, $\vec{v'}(AX) = \{1\}$. Thus $v'(\alpha \equiv \beta) = 1$ due to (1), which, according to (3) gives the result: $v(\alpha) = v(\beta)$, a contradiction with (2).

In order to prove that $V(\alpha) = V(\beta)$ consider any valuation $w \in Val_{SCI}$ determined by the homomorphism
$V : (L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv) \longrightarrow (P_{fin}(Var), id, \cup, \cup, \cup, \cup, \cup)$, i.e. for any formulas $\gamma, \delta \in L$, $w(\gamma \equiv \delta) = 1$ iff $w(\gamma) = w(\delta)$ and $V(\gamma) = V(\delta)$. So we obtain that $\vec{w}(AX) = \{1\}$. Hence and from (1) it follows that $w(\alpha \equiv \beta) = 1$ and consequently, $V(\alpha) = V(\beta)$.

As it could be expected, the first strengthening is stronger than the second one:

Theorem 3.

$\vdash_2 \subseteq \vdash_1$.