Several statements in algebraic logic, lattice theory and topology, that are closely related to Martin's axiom and the Baire Category Theorem are formulated. Provability and independence in ZFC of such statements are investigated.

We formulate several statements, addressing partially ordered sets, distributive lattices, boolean algebras, algebras studied in algebraic logic (like cylindric algebras), and certain topological spaces. Such statements are shown to be equivalent for any cardinal $\kappa \leq 2^{\aleph_0}$. Some of these equivalences were proved in [31].

$ZFC$ denotes Zermelo-Fraenkel set theory with choice and $CH$ is short for the continuum hypothesis. $MA$ is short for Martin's axiom. We do not assume $CH$ and for that matter the weaker $MA$. Axiom of choice is assumed unless otherwise explicitly specified. Supremum and infimum are denoted by $\sum$ and $\prod$ respectively. Throughout this paper $\kappa$ is a cardinal $\leq 2^{\aleph_0}$.

**Definition 1.** Let $A$ be a boolean algebra possibly with extra operations. $X \subseteq A$ is said to be a *non-principal* type if $\prod X = 0$.

Let $MA_\omega(\kappa)$ denote the following instance of $MA$. The subscript $\omega$ refers to the fact that we are considering only boolean algebras that are countable. Recall that $MA$ addresses the wider class of boolean algebras having the countable chain condition. $\kappa$ indicates the number of types considered.
\( MA_\omega(\kappa) \) Let \( A \) be a a countable boolean algebra. Let \( \{ X_i : i < \kappa \} \) be a family of non-principal types of \( A \). Then for every non zero \( a \in A \) there exists an ultrafilter \( F \) containing \( a \) such that \( F \cap -X_i \neq \emptyset \), where \( -X_i = \{-x : x \in X_i\} \). In forcing terminology, such an \( F \) is called a generic ultrafilter.

**Definition 2.** Let \( P = (P, \leq) \) be a partially ordered set. Let \( D \subseteq P \). Then \( D \) is dense in \( P \), if for all \( x \in P \), there is a \( d \in D \) such that \( d \leq p \).

An equivalent formulation of \( MA_\omega(\kappa) \) [5] addressing partially ordered sets is:

\( PO_\omega(\kappa) \) Let \( P \) be a countable partially ordered. Then for every family \( \{ D_i : i \in \kappa \} \) of dense sets in \( P \), there is a filter of \( P \) that meets every \( D_i \).

It is also known that \( MA_\omega(\kappa) \) is equivalent to \( RS(\kappa) \) where \( RS(\kappa) \) is short for the Rasiowa - Sikorski Lemma lifted to a cardinal \( \kappa \) that is not necessarily countable. In more detail:

\( RS(\kappa) \) Let \( B \) be a countable boolean algebra. Let \( J \) and \( K \) be sets such that \( |I| + |J| \leq \kappa \). Let \( \{ S_j : j \in J \} \) and \( \{ T_k : k \in I \} \) be families of subsets of \( B \) such that \( \sum S_j \) and \( \prod T_k \) exist for each \( j \in J \) and \( k \in K \). Then for every non-zero \( a \in B \), there exists an ultrafilter \( F \) containing \( a \) such that for all \( j \in J \) if \( \sum S_j \in F \) then \( S_j \cap F \neq \emptyset \) and for \( k \in K \) if \( T_k \subseteq F \) then \( \prod T_k \in F \).

Towards formulating a statement about Lattices we recall some notions.

**Definition 3.**

1. Let \( P = (P, \leq) \) be a partially ordered set. For elements \( x \) and \( y \) we say that \( x \) is below \( y \), in symbols \( x \ll y \), if for any filter \( D \) in \( P \) such that \( \sum D \) exists, if \( y \leq \sum D \), then there is some \( d \in D \) with \( x \leq d \).
2. \( P = (P, \leq) \) is upwards continuous if for all \( y \in P \) the set \( \{ x \in P : x \ll y \} \) is a directed set with supremum \( y \).
3. An element \( x \) in a lattice is a non-zero divisor if \( x \land y \neq 0 \) for any non-zero \( y \). An element \( x \) is irreducible if whenever \( x = a \land b \) then \( x = a \) or \( x = b \).

Now we are ready to formulate.
Variations on Martin’s Axiom and Omitting Types from Algebraic Logic...

Let $L = (L, \leq)$ be a countable distributive lattice that is upwards continuous. If $D$ is a set of $\kappa$ non-zero divisors of $L$, then there is an irreducible $p \in L$ such that no element of $D$ is below $p$.

We now turn to formulating several statements in algebraic logic. SC, QA, QEA and CA denote Pinter’s substitution algebras, Halmos’ quasipolyadic and quasipolyadic-equality algebras, and Tarski’s cylindric algebras, respectively. For $K \in \{SC, QA, QEA, CA\}$, and a countable ordinal $m$, $K_m$ denotes the algebras of dimension $m$. Such algebras are defined in e.g. [19] and [41].

For $m \leq \omega + \omega$ and $n < m$, $\mathfrak{Nr}_n K_m$ denotes the class of all $n$ neat reducts of algebras in $K_m$. The class of neat reducts is defined in [6], 2.6.29 for cylindric algebras; it generalizes verbatim to other algebras studied herein [41], [23], [24]. The notion of neat reducts is an old venerable notion in algebraic logic, that is gaining some momentum lately, cf. [20], [31], [22], [23], [7], [8], [26], [24], [18], [38], [32], [35], [36], [34], [40], [28], [33], [25], and [41]. For $B \in K_m$ and $n < m$, $\mathfrak{Nr}_n B$ denotes the neat $n$-reduct of $B$. $\mathfrak{Nr}_n B \in K_n$.

**Definition 4.** $A \in K_n$ has the strong neat embedding property if there exists $B \in K_{n+\omega}$ such that $A \subseteq \mathfrak{Nr}_n B$ and for all $X \subseteq A$, if $\sum^B X = 1$ then $\sum^A X = 1$.

Set algebras and weak set algebras are defined in [6]. We recall from op.cit that a set algebra of dimension $n$ has unit of the form $^n U$ and a weak set algebra of dimension $n$ has unit of the form $^n U(p)$ which is the set of sequences that agree cofinitely with $p \in ^n U$. In both cases, $U$ is called the base of the algebra.

Let $n$ be an ordinal $\leq \omega$. $R(n, K, \kappa)$ denote the following statement. $R(n, K, \kappa)$ is short for representing $K$ algebras of dimension $n$ such that $\kappa$ many types are omitted.

**$R(n, K, \kappa)$** Let $A \in K_n$ be countable. Assume that $A$ has the strong neat embedding property. Let $\{X_i : i < \kappa\}$ be a family of non-principal types. Then for every non zero $a \in A$ there exists a weak set algebra $E$ with countable base and a homomorphism $f : A \rightarrow E$ such that $f(a) \neq 0$ and $f$ omits the non principal types $X_i$ in the sense that for every $i < \kappa$, $\cap_{x \in X_i} f(x) = \emptyset$. 
LfK\_ω denotes the class of locally finite K algebras of dimension \(\omega\). \(\mathfrak A\) is locally finite if \(\Delta x = \{ i \in \omega : c_i x \neq x \} \) is finite for every \(x \in A\). LfK\_ω is the algebraic counterpart of first order logic [6] §4.3.

Let \(OT(\kappa)\) denote the following statement. \(OT(\kappa)\) is short for omitting \(\kappa\) many types.

\[ OT(\kappa) \quad \text{Let } \mathfrak A \in \text{LfK}_\omega \text{ be countable. Let } \{X_i : i < \kappa\} \text{ be a family of non-principal types. Assume further that for all } i \in \kappa \text{ there exists } n \in \omega \text{ such that } X_i \subseteq \mathfrak N r_n \mathfrak A. \text{ Then for every non zero } a \in A \text{ there exists a set algebra } \mathfrak C \text{ with countable base and a homomorphism } f : A \rightarrow \mathfrak C \text{ such that } f(a) \neq 0 \text{ and } \bigcap_{x \in X_i} f(x) = \emptyset \text{ for all } i \in \kappa. \]

We now recall a class of algebras introduced in [18] and further studied in [24]. We follow the terminology of [24]. Let \(\alpha\) be a countably infinite ordinal, and \(G \subseteq ^\omega \alpha\). Then the class of \(G\) polyadic set algebras, or \(\text{GPSA}_\alpha\), for short is defined in [24] 1.1. The class of abstract \(G\) algebras, or \(\text{GPA}_\alpha\), for short, is defined in [24] 2.2 via a finite schema of equations. Rich and strongly rich semigroups are defined in [24] 1.4. The main Theorem in [18] is that when \(G\) is a rich semigroup then the abstract and concrete \(G\) algebras coincide. (This is a completeness Theorem for the corresponding logic.) The main result in [24] is that when \(G\) happens to be strongly rich then \(G\) polyadic (set) algebras have the superamalgamation property. (This is an interpolation Theorem for the corresponding logic.) Now consider the statement \(OTG(\kappa)\) which refers to omitting \(\kappa\) many types in \(G\) algebras.

\[ OTG(\kappa) \quad \text{Let } G \text{ be a countable strongly rich sub-semigroup of } ^\omega \omega. \text{ Let } \mathfrak A \in \text{GPA}_\omega \text{ be countable. Let } \{X_i : i < \kappa\} \text{ be a family of non-principal types } \mathfrak A. \text{ Then for all non-zero } a \in A \text{, there exists } \mathfrak C \in \text{GPSA}_\omega \text{, with countable base, and a homomorphism } f : \mathfrak A \rightarrow \mathfrak C \text{ such that } f(a) \neq 0 \text{ and } \bigcap_{x \in X_i} f(x) = \emptyset \text{ for all } i \in \kappa. \]

In preparation of formulating our topological statements, we review some topological notions:

**Definition 5.**

1. Let \(X\) be a topological space and let \(\mathcal C\) be an open cover of \(X\). \(A \subseteq X\) is said to have diameter less than \(\mathcal C\), denoted by \(\delta(A) < \mathcal C\), if there is some \(U \in \mathcal C\) with \(A \subseteq U\).

2. Let \(\kappa\) be a cardinal. A space \(X\) is said to be \(\kappa\)-Čech complete if
(i) $X$ is regular, i.e every neighbourhood of a point $x \in X$ contains a closed neighbourhood of $x$.

(ii) There exists a collection $\{C_i : i \in \kappa\}$ of open covers of $X$ such that for any collection $\mathcal{F}$ of closed subsets of $X$ with the finite intersection property, if $\mathcal{F}$ contains, for each $i \in \kappa$, a set of diameter less than $C_i$ then $\mathcal{F}$ has non-empty intersection.

(3) A space is $\kappa$-Baire if the intersection of $\kappa$ many non-empty open dense sets is dense. A space is Baire if it is $\omega$-Baire.

Examples of Baire spaces, and Cech-complete spaces, are complete metric spaces. We refer the reader to [12] for more on such spaces. We recall that a topological space is second countable if it has a countable base.

Now let $\text{Cech}(\kappa)$ be the following statement.

$\text{Cech}(\kappa)$  Every second countable $\kappa$-Cech complete space is $\kappa$-Baire.

We formulate other topological properties related to the Baire property. We start by some definitions:

**Definition 6.** Let $X$ be a topological space:

1. $X$ is locally compact if every point has a compact neighbourhood.
2. If $U, V$ are subsets of $X$ then $U$ is compact in $V$ if every open cover of $V$ is reducible to a finite subcover of $U$.
3. $X$ is core compact if every open set $O$ is the union of open sets such that every such set is compact in $O$.
4. A subset $C$ of $X$ is irreducible if it is closed and the non empty open sets in the subspace topology on $C$ forms a filter base.
5. $X$ is sober if it is $T_0$ and every irreducible subset is the closure of a point.
6. A filter $F$ of open sets in $X$ is Scott open if whenever $D$ is a collection of open sets which is directed upward by inclusion $\bigcup D \subseteq F$, then some member of $D$ is contained in $F$.
7. $X$ is weakly $\kappa$-Baire if the intersection of $\kappa$ many open sets is non-empty.
Now let:

\textbf{Sober}(\kappa) \quad \text{Every sober locally compact second countable space is weakly } \kappa-\text{Baire.}

and

\textbf{Com}(\kappa) \quad \text{In a core compact second countable space, if } D \text{ is a collection of open dense sets, then there is an irreducible set meeting all elements of } D.

We shall now prove

\textbf{Theorem 8 (Main Theorem)}

(1) \text{The following statements are equivalent in } ZFC:\n
(i) \( (\forall \kappa) MA_\omega(\kappa). \)
(ii) \( (\forall \kappa) PO_\omega(\kappa). \)
(iii) \( (\forall \kappa) RS(\kappa). \)
(iv) \( (\forall \kappa) \text{Lat}_\omega(\kappa). \)
(v) \( (\forall \kappa)(\forall n \leq \omega)R(n, K, \kappa). \)
(vi) \( (\forall \kappa) OT(\kappa). \)
(vii) \( (\forall \kappa) \text{OTG}(\kappa). \)
(viii) \( (\forall \kappa) \text{Cech}(\kappa). \)
(ix) \( (\forall \kappa) \text{Sober}(\kappa). \)
(x) \( (\forall \kappa) \text{Com}(\kappa). \)

(2) \( (\forall \kappa) [MA_\omega(\kappa) \implies \kappa < 2^{\aleph_0}]. \)

(3) \( (\forall \kappa) [\kappa < 2^{\aleph_0} \implies MA_\omega(\kappa)] \text{ is independent from } ZFC + \neg CH. \)

(4) \text{Let } \text{cov}_K \text{ denote the least cardinal } \lambda \text{ such that } \mathbb{R} \text{ can be covered by } \lambda \text{ closed nowhere dense sets. Then } \text{cov}_K \text{ is the least cardinal } \kappa \text{ such that } MA_\omega(\kappa) \text{ is false. That is}

\( (\forall \kappa)[\kappa < \text{cov}_K \implies MA_\omega(\kappa)] \text{ is provable in } ZFC. \)

\textbf{Proof.} (2) (3) and (4) are proved in [31]. (i), (ii) (iii) (v), (vi) are proved to be equivalent in [31]. We note that (vi), (vii) and (viii) are algebraic versions of an Omitting Types theorems for variants of first order logic. Here we prove the rest of the equivalences.

- We start by \( PO_\omega(\kappa) \) implies \text{Sober}(\kappa). Let \((X, T)\) be a locally compact second countable sober space. Let \( \{A_i : \kappa \} \) be a given collection
of open sets. We want to show that \( \bigcap_{i \in \kappa} A_i \) is non-empty. Let \( \mathcal{B} \) be a countable base of \( X \). Define a partial order on \( \mathcal{B} \) by \( O_1 \leq O_2 \) if either \( O_1 \subseteq O_2 \) or \( O_1 \) is compact in \( O_2 \). By local compactness two open sets have an upper bound in \( (\mathcal{B}, \leq) \) if their intersection is non-empty.

For each \( i < \kappa \), let \( D_i = \{ O \in \mathcal{B} : O \subseteq A_i \} \).

By local compactness each \( D_i \) is dense in \( \mathcal{B} \). By \( \text{PO}_\omega(\kappa) \) there is a filter \( S \) of open sets that meets \( D_i \) for every \( i \). Suppose that \( \bigcap S \neq \emptyset \).

Then any element in \( \bigcap S \) is in some \( D_i \) and so is in \( \bigcap_{i \in \kappa} A_i \).

If \( S \) has a maximal element under \( \leq \) then it is in \( \bigcap S \) and we are done.

Otherwise \( S \) is Scott open since if \( D \) is a directed collection of open sets with \( \bigcup D \in S \) then some element in \( O \in S \) is strictly greater than \( \bigcup D \).

Then \( O \) is compact in \( \bigcup D \). We now use the Hofmann Mislove Theorem [10] which gives \( \bigcap S \neq \emptyset \). The Hofmann Mislove Theorem says that in a sober space the intersection of a Scott open filter is non-empty.

- **Sober(\( \kappa \)) implies Latt_\( \omega \)(\( \kappa \)).** Assume Sober(\( \kappa \)). Let \( \mathcal{L} \) be a countable upward continuous distributive lattice \( \mathcal{L} \). By Theorem 1.2 of [11] there exists a locally compact sober space \( X \) such that \( \mathcal{L} \) is order isomorphic to the lattice of open subsets of \( X \). Since \( \mathcal{L} \) is countable, \( X \) is second countable. Given a set \( \{ d_i : i \in \kappa \} \) of non-zero devisors the image of \( d_i \) under the isomorphism is an open dense subset \( O_i \) of \( X \).

So there is a point in \( \bigcap O_i \). Let \( O = X - \{ x \} \). Then \( O \) correspond to an irreducible \( p \) and since \( x \in O_i \) for all \( i \), we cannot have \( d_i \leq p \) for any \( i \).

- We now prove that Latt_\( \omega \)(\( \kappa \)) implies Com(\( \kappa \)). Let \( (X, T) \) be a second countable core compact space. Let \( \mathcal{B} \) be a countable base, Then \( (\mathcal{B}, \subseteq) \) is a countable upward continuous distributive lattice. Let \( \{ O_i : i \in \kappa \} \) be a collection of open dense sets. Each \( O_i \) is a non-zero devisor. So there is an irreducible \( O \) in the lattice which does not contain any \( O_i \). \( C = X - O \) is as desired. It is irreducible and meets every \( O_i \).

- Now assume \( R(n, K, \kappa) \). Take \( n = 0 \) and \( K = CA \) in \( R(n, K, \kappa) \).

Then MA_\( \omega \)(\( \kappa \)) follows. That OT(\( \kappa \)) and for that matter OTG(\( \kappa \)) implies MA_\( \omega \)(\( \kappa \)) is the same. In more detail, assume OT(\( \kappa \)). Let \( \mathfrak{A} \) be a boolean algebra. Give \( \mathfrak{A} \) the discrete structure by setting for all \( i, j < \omega \) and \( x \in A \), \( c_i x = x \) and \( d_{ij} = 1 \). Then \((\mathfrak{A}, c_i, d_{ij})_{i, j < \omega} \) is...
an $\mathbf{L}_\omega$. A representation preserving infinite meets in $\mathfrak{A}$ provided by $OT(\kappa)$ is nothing more than a boolean representation preserving the given meets.

- **Assume $Cech(\kappa)$.** We want $RS(\kappa)$. Let $\mathfrak{B}$ be the given countable Boolean algebra. Let $S_{\mathfrak{B}}$ be the space of ultrafilters of $\mathfrak{B}$, topologized by taking as as basic open sets the sets
  \[ S_b = \{ F \in S_{\mathfrak{B}} : b \in F \} \quad \text{for all } b \in B. \]
  This makes $S_{\mathfrak{B}}$ a second countable compact Hausdorff space (the Stone space of $\mathfrak{B}$). Then $S_{\mathfrak{B}}$ is $\kappa$-Baire, since the intersection of any collection of closed sets with the finite intersection property is non-empty. We say that $F$ preserves $X$ if $X \subseteq F$ implies that $\prod X \in F$. For $i \in \kappa$, put
  \[ S_i = \{ F \in S : F \text{ preserves } \prod X_i \}, \]
  it follows that
  \[ S_i = ( \bigcup_{x \in X_i} S_x') \cup S_{\prod X_i} \]
  so that $S_i$ is open. Moreover, $S_i$ is dense in $S_{\mathfrak{B}}$, for if $S_b$ is a non-empty basic open, then either
  (i) $b \leq \prod X_i$, whence $S_b \cap S_i = S_b \neq \emptyset$; or
  (ii) $b$ is not less than $x$ for some $x \in X_i$, whence $b \land x' \neq 0$, so any $F \in S_{\mathfrak{B}}$ with $b \land x' \in F$ has $F \in S_b \cap S_i$. In either case then, we get $S_b \cap S_i \neq \emptyset$ as desired. It follows, since $S_{\mathfrak{B}}$ is a Baire space, that the set $\bigcap_{i \in \kappa} S_i$ of ultrafilters that preserves all meets $\prod X_i$ is dense in $S_{\mathfrak{B}}$. But then if $a \neq 0$, $S_a$ is non-empty and open, so intersects $\bigcap_{i \in \kappa} S_i$.

- **Assume $Com(\kappa)$.** We prove $PO\omega(\kappa)$. Let $(P, \leq)$ be as in the hypothesis of $PO\omega(\kappa)$. Consider the partial order topology $T$ generated by all sets of the form
  \[ O_x = \{ y : x \leq y \}. \]
  $T$ is second countable and locally compact, hence core compact. Given a collection $\{D_i : i \in \kappa\}$ of dense subsets in $T$ there is an irreducible $C$ meeting all these sets. $C$ is directed, hence we are done.

- **Finally, assume $RS(\kappa)$.** We now prove $Cech(\kappa)$. Towards this end, let $S$ be a Cech complete second countable space with associated open
covers $C_i$ for $i \in \kappa$. Let $\{S_i, i \in \kappa\}$ be a collection of open dense subsets of $S$. We want to show that $\bigcap_{i \in \kappa} S_i$ is dense. Let $\mathcal{B}$ be the boolean algebra of regular open subsets of $S$, a regular open set being one equal to the interior of its closure. Then $\mathcal{B}$ is a complete boolean algebra with a countable dense subset. Put for each $i \in \kappa$

$$X_i = \{U' : U \in B, \bar{U} \subseteq S_i, \text{ and } \delta(\bar{U}) < C_i\}.$$  

Here $U'$ denotes the boolean complement of $U$ in $B$, i.e. the interior of $S \setminus U$, and $\bar{U}$ denotes the topological closure of $U$ in $B$. The meet of $X_i$ in $B$ is the interior of $\bigcap X_i$. But if $W$ is a non-empty open subset of $\bigcap X_i$, density of $S_i$ implies that there is some $x \in W \cap S_i$. As $C_i$ is a cover, there is then some $C \in C_i$ with $x \in C$. By regularity of $S$, $x$ has a closed neighbourhood $V \subseteq W \cap S_i \cap C$. Taking $U$ as the interior of $V$ gives $x \in U \in B$, $\bar{U} \subseteq S_i$ and $\delta(\bar{U}) < C_i$. Hence $U' \in X_i$. Since $x \in W \subseteq \bigcap X_i$, this gives $x \in U'$, contrary to the fact that $x \in U$. Thus $\bigcap X_i$ has empty interior, i.e. $\prod X_i = 0$ in $\mathcal{B}$.

Now to show that $\bigcap_{i \in \kappa} S_i$ is dense. Let $W$ be any non-empty open set, and apply regularity to get a non-empty $U \in B$ with $\bar{U} \subseteq W$. By $RS(\kappa)$ there is a ultrafilter $F$ of $B$ with $U \in F$, such that $F$ preserves $\prod X_i$ for all $i$, in other words if $X_i \subseteq F$ then $\prod X_i \in F$.

Let $\mathcal{F} = \{V : V \in F\}$. Then $\mathcal{F}$ has the finite intersection property since $F$ does. Also for each $i \in \kappa$, since $\prod X_i = 0$ and $F$ preserves $\prod X_i$, there is some member of $X_i$ whose $B$ complement is in $F$, i.e there is some $U_i$ with $\bar{U}_i \subseteq S_i$, $\delta(\bar{U}_i) < C_i$ and $\bar{U}_i \in F$; hence $\bar{U}_i \in \mathcal{F}$.

In other words, $\mathcal{F}$ contains sets of diameter less than $C_i$, for all $i$, so by Cech-completeness of $S$, there exists $x \in \cap \mathcal{F}$. In particular, there exists $x \in \bar{U} \subseteq W$, and, for each $i$, taking a $U_i$ as above gives $x \in U_i \subseteq S_i$. Hence

$$\left(\bigcap_{i \in \kappa} S_i\right) \cap W \neq \emptyset.$$  

We are done. \hfill \Box

References


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