Andrei Khrennikov, Andrew Schumann

LOGICAL APPROACH TO \( p \)-ADIC PROBABILITIES

Abstract
In this paper we considered a moving from classical logic and Kolmogorov’s probability theory to non-classical \( p \)-adic valued logic and \( p \)-adic valued probability theory. Namely, we defined \( p \)-adic valued logic and further we constructed probability space for some ideals on truth values of \( p \)-adic valued logic. We proposed also \( p \)-adic valued inductive logic. Such a logic was considered for the first time. The main originality of \( p \)-adic valued inductive logic consists in the non-classical interpretation of the negation symbol.

The standard definition of probabilities that are usually assumed to be real numbers is Kolmogorov’s definition. His probability theory is reduced to the theory of normalized \( \sigma \)-additive measures taking values in the segment \([0, 1]\) of the field of real numbers. Non-Kolmogorovian probabilistic models for \( p \)-adic quantum physics were proposed in [4], [5].

In this paper, \( p \)-adic probabilities are constructed on the base of \( p \)-adic valued logic. The building of \( p \)-adic valued logic allows to set a logical lattice for \( p \)-adic probabilities. Therefore it is possible also to construct \( p \)-adic inductive logic (\( p \)-adic probability logic).

Recall that in deductive logic the syntactic structure of the sentences involved completely determines whether premises logically entail a conclusion. In inductive logic each sentence confers a syntactically specified degree of support on each of the other sentences of the language. The inductive probabilities in such a system are logical in the sense that they depend on syntactic structure alone. This kind of conception was first articulated by John Maynard Keynes in [2] and was developed by Rudolf Carnap in [1].
The main originality of \( p \)-adic valued inductive logic consists in the other interpretation of the complement (negation) that isn’t Boolean.

Let us remark that \( p \)-adic physics (see [9], [10], [3], [4], [5]) was based on the change of space paradigm, on the moving from the continuous real space to discrete-like \( p \)-adic space. We found that this change of space paradigm is coupled to the change in logical and probabilistic paradigms: from Boolean logic and Kolmogorov’s probability to non-Boolean \( p \)-adic valued inductive logic and \( p \)-adic valued probability.

Let us remember that the expansion

\[
n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=0}^{\infty} \alpha_k \cdot p^k,
\]

where \( \alpha_k \in \{0, 1, \ldots, p - 1\} \), \( \forall k \in \mathbb{N} \), is called the expansion of \( p \)-adic integer \( n \). The integer \( n \) is called \( p \)-adic. This number sometimes has the following notation: \( n = \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0 \). We denote the set of such numbers by \( \mathbb{Z}_p \).

Extend the standard order structure on \( \{0, \ldots, p - 1\} \) to a partial order structure on \( \mathbb{Z}_p \). Define this partial order structure on \( \mathbb{Z}_p \) as follows:

\( \mathcal{O}_{\mathbb{Z}_p} \) Let \( x = \ldots x_n \ldots x_1 x_0 \) and \( y = \ldots y_n \ldots y_1 y_0 \) be the canonical expansions of two \( p \)-adic integers \( x, y \in \mathbb{Z}_p \). We set \( x \leq y \) if we have \( x_n \leq y_n \) for each \( n = 0, 1, \ldots \). We set \( x < y \) if we have \( x_n \leq y_n \) for each \( n = 0, 1, \ldots \) and there exists \( n_0 \) such that \( x_{n_0} < y_{n_0} \). We set \( x = y \) if \( x_n = y_n \) for each \( n = 0, 1, \ldots \).

Now introduce two operations \( \sup, \inf \) in the partial order structure on \( \mathbb{Z}_p \):

1. For all \( p \)-adic integers \( x, y \in \mathbb{Z}_p \), \( \inf(x, y) = x \) if and only if \( x \leq y \) under condition \( \mathcal{O}_{\mathbb{Z}_p} \). For all \( p \)-adic integers \( x, y \in \mathbb{Z}_p \), \( \sup(x, y) = y \) if and only if \( x \leq y \) under condition \( \mathcal{O}_{\mathbb{Z}_p} \).
2. Let \( x = \ldots x_n \ldots x_1 x_0 \) and \( y = \ldots y_n \ldots y_1 y_0 \) be the canonical expansions of two \( p \)-adic integers \( x, y \in \mathbb{Z}_p \) and \( x, y \) are incompatible under condition \( \mathcal{O}_{\mathbb{Z}_p} \). We get \( \inf(x, y) = z = \ldots z_n \ldots z_1 z_0 \), where, for each \( n = 0, 1, \ldots \), we set (1) \( z_n = y_n \) if \( x_n \geq y_n \), (2) \( z_n = x_n \) if \( x_n \leq y_n \), (3) \( z_n = x_n = y_n \) if \( x_n = y_n \). We get \( \sup(x, y) = z = \ldots z_n \ldots z_1 z_0 \), where, for each \( n = 0, 1, \ldots \), we set (1) \( z_n = y_n \) if \( x_n \leq y_n \), (2) \( z_n = x_n \) if \( x_n \geq y_n \), (3) \( z_n = x_n = y_n \) if \( x_n = y_n \).
It is important to remark that there exists the maximal number \( N_{\max} \in Z_p \) under condition \( \mathcal{O}_{Z_p} \). It is easy to see: \( N_{\max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots \).

Consider a first-order language \( \mathcal{L} \) that is built in the standard way. Let this language be associated with the following matrix \( M_{Z_p} = <V_{Z_p}, \neg, \lor, \land, \exists, \forall, \{N_{\max}\}> \) that is called \( p \)-adic valued matrix logic, where (1) \( V_{Z_p} = \{0, \ldots , N_{\max}\} = Z_p \), (2) for all \( x \in V_{Z_p}, \neg x = N_{\max} - x \), (3) for all \( x, y \in V_{Z_p}, x \lor y = (N_{\max} - \text{sup}(x, y)) \), (4) for all \( x, y \in V_{Z_p}, x \land y = \neg(\neg x \lor \neg y) = \inf(x, y) \), (6) for a subset \( M \subseteq V_{Z_p}, \exists(M) = \sup(M) \), where \( \sup(M) \) is a maximal element of \( M \), (7) for a subset \( M \subseteq V_{Z_p}, \forall(M) = \inf(M) \), where \( \inf(M) \) is a minimal element of \( M \), (8) \( \{N_{\max}\} \) is the set of designated truth values.

**Proposition 1.** Let \( M_{Z_p} = <V_{Z_p}, \neg, \lor, \land, \exists, \forall, \{N_{\max}\}> \) be a \( p \)-adic valued matrix logic. Restrict the set \( V_{Z_p} \) to the set \( V \) that contains only \( p \)-adic integers satisfying the following property: if \( \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0 \) is the expansion of \( p \)-adic integer \( x \in V \), then for each \( j = 0, 1, 2, \ldots \) we have \( \alpha_j \in \{0, p-1\} \subseteq \{0, \ldots , p-1\} \).

Then a restricted matrix logic \( M_V = <V, \neg, \lor, \land, \exists, \forall, \{N_{\max}\}> \) is a Boolean algebra.

**Corollary 1.** The logic \( M_{Z_2} = <V_{Z_2}, \neg, \lor, \land, \exists, \forall, \{N_{\max}\}> \) is a Boolean algebra.

A nonempty subset of truth values \( \Delta \subseteq V_{Z_p} \) is said to be an ideal on truth values of \( M_{Z_p} \) if the following condition holds: for all \( x, y \in Z_p \), \( \sup(x, y) \in \Delta \) iff \( x \in \Delta \) and \( y \in \Delta \). A subset of truth values \( \Delta \subseteq V_{Z_p} \) is called a complete ideal if \( \Delta \) is defined as follows: (1) \( \Delta \) is an ideal, (2) \( \Delta \neq \{0\} \), (3) if \( N \) is a maximal truth value that is contained in \( \Delta \), then \( \Delta \) contains all truth values that are smaller than \( N \). We shall say that an ideal \( \Delta \) is premaximal if \( \Delta \) is complete and its maximal member \( N \) has the following property: if \( \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0 \) is the expansion of \( N \), then for each \( j = 0, 1, 2, \ldots \) we have \( \alpha_j \in \{0, p-1\} \subseteq \{0, \ldots , p-1\} \).
Proposition 2. Complete ideals are either mutually disjoint (only \( \{0\} \) is their common member) or they are contained in another.

Since a complete ideal \( \Delta \subseteq \mathbb{Z}_p \) contains the maximal \( p \)-adic integer \( N_{\text{max}} \), we see that \( \Delta \) is equal to \( V_{\mathbb{Z}_p} \).

Proposition 3. Let \( S \) be the family of all premaximal ideals on truth values of \( M_{\mathbb{Z}_p} \). Consider the algebraic system \( F = < S \cup \{0\} \cup V_{\mathbb{Z}_p}, \neg, \cap, \cup > \), where (1) \( S \cup \{0\} \cup V_{\mathbb{Z}_p} \) is the domain of \( F \), (2) for any \( \Delta \in S \), if \( N \) is the maximal member of \( \Delta \), then by \( \neg \Delta \) we denote a premaximal ideal that contains the maximal member \( \neg N \), (3) for any \( \Delta_1, \Delta_2 \in S \), \( \Delta_1 \cap \Delta_2 \) is the intersection of the sets \( \Delta_1, \Delta_2 \), (4) for any \( \Delta_1, \Delta_2 \in S \), \( \Delta_1 \cup \Delta_2 \) is the union of the sets \( \Delta_1, \Delta_2 \) such that \( \Delta_1 \cup \Delta_2 \) contains also least upper bounds of all truth values which belong to \( \Delta_1, \Delta_2 \) respectively.

Then the algebraic system \( F \) is a Boolean algebra.

Now show that the algebraic system \( F \) can be transformed to a \( p \)-adic probability space. Let \( |A| \) be a cardinality of a set \( A \). Define the new operation:

\[
||A|| = \begin{cases} 
||A|| = 0 & \text{if } |A| = 1; \\
||A|| = |A| & \text{otherwise.}
\end{cases}
\]

It is easy to check that if \( \Delta \) is a premaximal ideal and \( N \) is its maximal element, then \( ||\Delta|| = |\Delta| = N \).

Consider some premaximal ideal \( \Delta = \Delta_N \), for which \( N \) is a maximal \( p \)-adic integer that is contained in \( \Delta \). This means that the cardinality \( ||\Delta|| = N \). We define the probability of a premaximal ideal \( A \) by the standard proportional relation:

\[
P(\Delta) \triangleq P_{\Delta_N}(A) = \frac{||A \cap \Delta_N||}{||\Delta_N||} = \frac{||A \cap \Delta_N||}{N}. \tag{1}
\]

Notice that the probabilities of the sets \( \{0\} \) and \( V_{\mathbb{Z}_p} \) are defined as follows:

\[
P(\{0\}) = \frac{||\{0\} \cap \Delta_N||}{||\Delta_N||} = \frac{||\{0\}||}{N} = 0, \tag{2}
\]

\[
P(V_{\mathbb{Z}_p}) = \frac{||V_{\mathbb{Z}_p} \cap \Delta_N||}{||\Delta_N||} = \frac{||\Delta_N||}{N} = 1. \tag{3}
\]
Denote the family of all subsets $A \subset (S \cup \{0\} \cup V_{\mathbb{Z}_p})$, for which $P(A)$ exists, by $G_\Delta$. The sets $A \in G_\Delta$ are said to be events.

**Proposition 4.** $G_\Delta$ consists of all subsets and their complements.

**Proposition 5.** Let $A_1, A_2 \in G_\Delta$ and $A_1 \cap A_2 = \{0\}$. Then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

**Proposition 6.** Let $A_1, A_2 \in G_\Delta$. Then $G_\Delta$ contains the following sets:

1. $A_1 \cup A_2 \in G_\Delta$,
2. $A_1 \cap A_2 \in G_\Delta$,
3. $A_1 \setminus A_2 = A_1 \cap \neg A_2 \in G_\Delta$,
4. $A_2 \setminus A_1 = A_2 \cap \neg A_1 \in G_\Delta$,

where the operations $\cap$, $\cup$, and $\neg$ are defined in proposition 3.

**Corollary 2.** The family $G_\Delta$ is an algebra of subsets.

There are standard formulas for $A_1, A_2 \in G_\Delta$:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2);$$

$$P(A_1 \setminus A_2) = P(A_1) - P(A_1 \cap A_2).$$

The ordered system $<\Delta_N, G_\Delta, P_{\Delta_N}>$ is called a $p$-adic probability space for the premaximal ideal $\Delta_N$.

**Proposition 7.** (Conditional probability) Take a premaximal ideal $A \in G_\Delta$ such that $A \subset \Delta_N$ and let $B \in G_A$. Then $B \in G_\Delta$ and Bayes formula

$$P_A(B) = P_{\Delta}(B/A) = \frac{P_{\Delta}(B \cap A)}{P_{\Delta}(A)}$$

is valid.

**Proposition 8.** Consider the set $V_{\mathbb{Z}_p}$. It is the largest set of the algebraic system $F$: $||V_{\mathbb{Z}_p}|| = N_{\text{max}} = -1$. Then the following statement are true:

1. for any premaximal ideal $\Delta_N$, we obtain $\Delta_N \in G_{V_{\mathbb{Z}_p}}$ and

$$P_{V_{\mathbb{Z}_p}}(\Delta_N) = \frac{||\Delta_N||}{||V_{\mathbb{Z}_p}||} = -N$$
Andrei Khrennikov, Andrew Schumann

\( \mathcal{G}_\Delta \subset \mathcal{G}_{V_{Z_p}} \) and probabilities \( P_\Delta(A) \) are calculated as conditional probabilities with respect to the subsets \( \Delta_N \) of the set \( V_{Z_p} \):

\[
P_\Delta(A) = P_{V_{Z_p}}(A/\Delta_N) = \frac{P_{V_{Z_p}}(A \cap \Delta_N)}{P_{V_{Z_p}}(\Delta_N)}.
\]

Now consider the algebraic system \( F' = \langle S \cup \{0\} \cup V_{Z_p}, \cap, \cup, \rangle \), where

1. \( S \) is the set of all complete ideals,
2. \( S \cup \{0\} \cup V_{Z_p} \) is the domain of \( F' \),
3. for any \( \Delta_1, \Delta_2 \in S \), \( \Delta_1 \cap \Delta_2 \) is the intersection of the sets \( \Delta_1, \Delta_2 \),
4. for any \( \Delta_1, \Delta_2 \in S \), \( \Delta_1 \cup \Delta_2 \) is the union of the sets \( \Delta_1, \Delta_2 \) such that \( \Delta_1 \cup \Delta_2 \) contains also least upper bounds of all truth values which belong to \( \Delta_1, \Delta_2 \) respectively.

Let \( \Delta = \Delta_N \) be a complete ideal, for which \( N \) is a maximal \( p \)-adic integer that is contained in \( \Delta \). Then we can define the probability of a complete ideal \( A \) as follows:

\[
P(A) \triangleq P_{\Delta_N}(A) = \frac{||A \cap \Delta_N||}{||\Delta_N||} = \frac{||A \cap \Delta_N||}{N}.
\]

The probabilities of the sets \( \{0\} \) and \( V_{Z_p} \) have the notations (2) and (3) respectively.

Denote the family of all subsets \( A \subset (S \cup \{0\} \cup V_{Z_p}) \) of \( F' \), for which \( P(A) \) exists, by \( \mathcal{G}_\Delta \). The sets \( A \in \mathcal{G}_\Delta \) are said to be events.

**Proposition 9.** Let \( A_1, A_2 \in \mathcal{G}_\Delta \). Then \( \mathcal{G}_\Delta \) contains the following sets:

1. \( A_1 \cup A_2 \in \mathcal{G}_\Delta \),
2. \( A_1 \cap A_2 \in \mathcal{G}_\Delta \).

The ordered system \( \langle \Delta_N, \mathcal{G}_\Delta, P_{\Delta_N} \rangle \) is called a \( p \)-adic positive probability space for the complete ideal \( \Delta_N \).

We shall say that a formula \( \varphi \) of a logical language \( \mathcal{L} \) has the truth valuation \( v(\varphi) = 0 \in Z_p \) in \( M_{Z_p} \) if \( \varphi \) has the truth value 0 (‘false’), a formula \( \varphi \in \mathcal{L} \) has the truth valuation \( v(\varphi) = N_{\text{max}} \in Z_p \) in \( M_{Z_p} \) if \( \varphi \) has the truth value \( N_{\text{max}} \) (‘true’), a formula \( \varphi \) of a logical language \( \mathcal{L} \) has the truth valuation \( v(\varphi) = x \in Z_p \) in \( M_{Z_p} \) if \( \varphi \) has the truth value \( x \) (‘neutral’).

A function \( P(\varphi) \) is said to be a probability measure of a formula \( \varphi \) in \( M_{Z_p} \) if \( P(\varphi) \) ranges over the number set \( Q_p \) and satisfies the following axioms:
1. $P(\varphi) = P(v(\varphi)) = \frac{v(\varphi)}{N_{\text{max}}}$, where $v(\varphi)$ is a truth valuation of $\varphi$;
2. if a conjunction $\varphi \land \psi$ has the truth valuation 0, then $P(\varphi \lor \psi) = P(\varphi) + P(\psi)$,
3. $P(\varphi \land \psi) = \min(P(\varphi), P(\psi))$.

Notice that taking into account condition 1 of our definition, if $\varphi$ has the truth value $v(\varphi) = N_{\text{max}}$ for any truth valuations, i.e., $\varphi$ is a tautology, then $P(\varphi) = 1$ in all possible worlds, and if $\varphi$ has the truth value $v(\varphi) = 0$ for any truth valuations, i.e., $\varphi$ is a contradiction, then $P(\varphi) = 0$ in all possible worlds. Under condition 2, we obtain $P(\neg \varphi) = 1 - P(\varphi)$.

Since $P(N_{\text{max}}) = 1$, we have
$$P(\sup\{x \in V_{Z_p}\}) = \sum_{x \in V_{Z_p}} P(x) = 1$$

By propositions 8, all events have a conditional plausibility:
$$P(\varphi) \equiv P_{N_{\text{max}}} (v(\varphi)) = P(v(\varphi)/N_{\text{max}}).$$

Now define the measure $P(\varphi/\psi)$ of the plausibility of proposition $\varphi$ conditional on the information stated in proposition $\psi$:
$$P(\varphi/\psi) = \frac{P(\varphi \land \psi)}{P(\psi)} = \frac{P(v(\varphi) \land v(\psi))}{P(v(\psi))},$$
where $v(\varphi)$ is a truth valuation of $\varphi$ and $v(\psi)$ is a truth valuation of $\psi$.

This measure is called a support function. One reading is to take each $P$ as a measure on possible worlds, or possible states of affairs. The idea is that, given a fully meaningful language $L$, $P(\varphi/\psi) = x \in Q_p$ says that among the worlds in which $\psi$ is true, $\varphi$ is true in proportion $x$ of them.

**P1** If $v(\varphi) = 0$ for any truth valuations, then $P(\varphi/\psi) = 0$ for all sentences $\psi$.

**P2** If $\varphi$ is a tautology, i.e., $v(\varphi) = N_{\text{max}}$ for any truth valuations, then we have $P(\varphi/\psi) = 1$ for all sentences $\psi$.

**P3** If $P(\varphi/\psi) = 1$ for all sentences $\psi$, then $\varphi$ is a tautology.

**P4** If $v(\psi \supset \varphi) = N_{\text{max}}$ for any truth valuations, then $P(\varphi/\psi) = 1$.

Indeed, in this case $P(\varphi/\psi) = \frac{P(v(\varphi) \land v(\psi))}{P(v(\psi))} = \frac{P(v(\psi))}{P(v(\psi))} = 1$.

**P5** If $v(\psi \equiv \varphi) = N_{\text{max}}$ for any truth valuations, then $P(\chi/\psi) = P(\chi/\varphi)$.

**P6** $P(\varphi/\psi \land \varphi) = 1$. 

It is obvious, $P(\varphi/\psi \land \varphi) = P(\varphi \land \varphi)/P(\varphi) = P(\varphi)/P(\varphi) = 1$.

P7 If $v(\psi \land \varphi) = 0$ for any truth valuations, then $P(\psi \lor \varphi/\chi) = P(\psi/\chi) + P(\varphi/\chi)$.

P8 If $\varphi_1, \varphi_2, \ldots, \varphi_N$ are mutually exclusive, then $P(\varphi_1 \lor \varphi_2 \lor \ldots \lor \varphi_N/\chi) = \sum_{n=1}^{N} P(\varphi_n/\chi)$.

It follows from the above mentioned rule by principle of mathematical induction.

P9 $P(\varphi \land \psi/\chi) = P(\varphi/\psi \land \chi) \cdot P(\psi/\chi)$.

Indeed, $P(\varphi \land \psi/\chi) = \frac{P(\varphi \land \psi \land \chi)}{P(\chi)} = \frac{P(\varphi \land \psi \land \chi)}{P(\psi \land \chi)} = P(\varphi/\psi \land \chi) \cdot P(\psi/\chi)$.

Notice that all formulas of real inductive logic are also valid in $p$-adic inductive logic if we restrict the set $\mathbb{Z}_p$ of $p$-adic integers to the set $\mathcal{V}$ that contains only $p$-adic integers satisfying the following property: if $\ldots \alpha_3\alpha_2\alpha_1\alpha_0$ is the expansion of $p$-adic integer $x \in \mathcal{V}$, then for each $j = 0, 1, 2, \ldots$ we have $\alpha_j \in \{0, p-1\} \subseteq \{0, \ldots, p-1\}$.

Under this condition, we have the Boolean complement. For example, the formula $P(\varphi/\psi) + P(\neg \varphi/\psi) = 1$ is valid for $p$-adic integers of $\mathcal{V}$, but in general case, it isn’t valid in $p$-adic inductive logic. Therefore $p$-adic inductive logic is more general probability logic than real one.

References


Andrei Khrennikov Director of International Center for Mathematical Modeling in Physics, Engineering, Economy and Cognitive Sc., University of Växjö, Sweden
e-mail: Andrei.Khrennikov@vxu.se

Andrew Schumann Department of Philosophy and Science Methodology, Belarusian State University, Minsk, Belarus
e-mail: Andrei.Schumann@uni.torun.pl