EXTREMAL CONSEQUENCE OPERATIONS

Abstract

Dzik [2] gives a direct proof of the axiom of choice from the generalized Lindenbaum extension theorem LET. Inspection of Dzik’s proof shows that its premise LET assumes only that LET holds for those consequence operations, here called extremal, for which \( Cn(Y) = Y \) or \( Cn(Y) = S \). A direct proof, not using the axiom of choice, is given for the conditional \( \text{Let} \Rightarrow \text{LET} \).

1. Introduction

Let \( S \) be any set. According to the standard postulates of Tarski, an operation \( Cn: \wp(S) \rightarrow \wp(S) \) is a consequence operation if for all \( X \subseteq S \):

\[
\begin{align*}
(a) & \quad X \subseteq Cn(X) \subseteq S \\
(b) & \quad Cn(Cn(X)) = Cn(X) \\
(c) & \quad Cn(X) = \bigcup \{ Cn(Z) \mid Z \subseteq X \text{ and } |Z| < \aleph_0 \}.
\end{align*}
\]

(1)

Tarski proposed a further postulate stating that \( S \) is denumerable, but that postulate is deliberately excluded here. (1c) is a set-theoretical combination of the principles of monotony (1c\(_0\)) and finitariness (1c\(_1\)):

\[
\begin{align*}
(c_0) & \quad \text{if } Z \subseteq X \text{ then } Cn(Z) \subseteq Cn(X) \\
(c_1) & \quad \text{if } y \in Cn(X) \text{ then } y \in Cn(Z) \text{ for some finite } Z \subseteq X.
\end{align*}
\]

(1)
\(\langle S, \mathsf{Cn} \rangle\) is a (deductive) calculus if it obeys (1). (Reference to \(S\) may be omitted.) \(Y\) implies \(y\) if \(y \in \mathsf{Cn}(Y)\). For all \(S\), the operations \(\mathsf{Cn}_{\text{min}}\) (= \(Y\) for all \(Y \subseteq S\)) and \(\mathsf{Cn}_{\text{max}}\) (= \(S\) for all \(Y \subseteq S\)) obey (1).

The set \(Y \subseteq S\) is a theory if \(\mathsf{Cn}(Y) = Y\) and is inconsistent if \(\mathsf{Cn}(Y) = S\). By (1a), \(S\) is a theory. \(\mathsf{Cn}(Y)\) is maximal if it is consistent and no proper extension is consistent. More generally, \(\mathsf{Cn}(Y)\) is \(y\)-saturated if it does not imply \(y\), and each proper extension implies \(y\):

\[
\begin{align*}
(a) \quad & y \notin \mathsf{Cn}(Y) \\
(b) \quad & y \in \mathsf{Cn}(Y \cup \{x\}) \text{ for each } x \notin \mathsf{Cn}(Y).
\end{align*}
\]

(2)

It is easily checked that every maximal theory \(Y\) is \(y\)-saturated for every \(y \notin Y\) (Lemma 7 below).

\(\langle S, \mathsf{Cn} \rangle\) is a compact calculus if every inconsistent \(Y \subseteq S\) has a finite inconsistent subset. The most familiar version \(\text{LT}\) of the Lindenbaum extension theorem states that if \(\langle S, \mathsf{Cn} \rangle\) is compact then every consistent \(Y \subseteq S\) has a maximal extension. Compactness is not a necessary condition for the validity of this theorem, as shown by the example of \(\langle S, \mathsf{Cn}_{\text{min}} \rangle\) when \(S\) is infinite, but there do exist calculi for which \(\text{LT}\) fails that are not compact. A simple example is provided by setting \(S\) equal to \(\mathcal{N}\), the set of natural numbers, and \(C(Y) = \{i \in \mathcal{N} \mid \exists k \in Y (i \leq k)\}\) for each \(Y \subseteq \mathcal{N}\). The Lindenbaum theorem \(\text{LT}\) fails in this calculus, since there is no maximal subset \(Y \subset \mathcal{N}\).

The generalized Lindenbaum extension theorem \(\text{LET}\), which holds for any calculus, compact or not, states that if \(y \notin \mathsf{Cn}(Y)\), then \(\mathsf{Cn}(Y)\) has a \(y\)-saturated extension. Löb [4] (pp. 238f.) gives an early formulation and proof of \(\text{LET}\), and it has been formulated also by Asser, da Costa, and others. The well known proofs of \(\text{LT}\) and \(\text{LET}\), which are similar, appeal crucially to the axiom of choice \(\text{AC}\), usually in the form of Zorn’s lemma or of the well-ordering theorem.

Dzik [2] gives a direct proof that each of \(\text{LT}\) and \(\text{LET}\) also implies \(\text{AC}\). If \(J\) is any family of non-empty sets, put \(S = \{\langle Y, y \rangle \mid y \in Y \in J\}\). The operation \(\mathsf{Cn} : \varphi(S) \mapsto \varphi(S)\) defined in (3) is plainly a compact consequence operation:

\[
\mathsf{Cn}(Y) = \begin{cases} 
Y & \text{if } Y \text{ is a function} \\
S & \text{otherwise.}
\end{cases}
\]

(3)
If every $Y \in J$ is a unit set, $S$ itself is a choice function for $J$. If not, choose some $Y \in J$ for which $|Y| > 1$, and some $y \in Y$. Since $y \not\in Cn(\emptyset)$, by LET (or LT) there is a $y$-saturated (or maximal) theory $Cn(X)$ that extends $\emptyset$. It is straightforward to show that $Cn(X)$ is a choice function for $J$.

2. Extremal Consequence Operations

If the operations $C_{n_{\text{min}}}$ and $C_{n_{\text{max}}}$ defined above are called extreme, those operations $Cn$ for which $Cn(Y) = Y$ or $Cn(Y) = S$ may be called extremal. Let Let and Lt be the restrictions of LET and LT to extremal calculi. Dzik’s proof, combined with the usual proofs of the unrestricted theorems LET and LT, establishes that the conditionals $\text{Let} \Rightarrow \text{LET}$ and $\text{Lt} \Rightarrow \text{LT}$ are set-theoretically valid. In this section we shall show that $\text{Let} \Rightarrow \text{LET}$ is open to a direct proof that does not appeal to AC.

Let $(S, Cn)$ be a calculus and $y$ an element of $S$. We write $L(Cn, y)$ for the statement that every $Y \subseteq S$ not implying $y$ can be extended in $(S, Cn)$ to a $y$-saturated theory: that is,

- for every $Y$, if $y \not\in Cn(Y)$ then there is $X \supseteq Y$ such that
  - (a) $y \notin Cn(X) = X$
  - (b) $y \in Cn(X \cup \{x\})$ for every $x \notin X$.

In these terms, what LET asserts is that $L(Cn, y)$ holds for every calculus $(S, Cn)$ and every element $y$ of $S$.

What will be shown here is that each consequence operation $Cn$ may be associated with a family of explicitly defined extremal consequence operations $\{Cn^{y} \mid y \in S\}$ such that for every $y$, $L(Cn^{y}, y)$ if and only if $L(Cn, y)$. It follows that LET holds generally if it holds for every extremal consequence operation.

Let $Cn$ be a (finitary) consequence operation and $y$ an element of $S$. The consequence operation $Cn^{y}$ is defined as follows.

$$Cn^{y}(Y) = \begin{cases} 
Y & \text{if } y \notin Cn(Y) \\
S & \text{otherwise}.
\end{cases}$$

**Theorem 1.** Let $(S, Cn)$ be a deductive calculus. For each $y$ the operation
\( \text{Cn}^y \) defined in (5) is an extremal consequence operation.

**Proof.** It is obvious that \( Y \subseteq \text{Cn}^y(Y) = \text{Cn}^y(\text{Cn}^y(Y)) \), and that \( \text{Cn}^y \) is extremal. Assume that \( X \subseteq Z \) and that \( z \notin \text{Cn}^y(Z) \). By (5), \( \text{Cn}^y(Z) = Z \) and \( y \notin \text{Cn}(Z) \). Hence \( z \notin Z \) and (by \( \text{Cn} \) obeys (1c)) \( y \notin \text{Cn}(X) \). Hence (by assumption) \( z \notin X \) and (by (5) once more) \( \text{Cn}^y(X) = X \). It follows that \( z \notin \text{Cn}^y(X) \). This proves that \( \text{Cn}^y \) obeys monotony (1c).

To show that \( \text{Cn}^y \) is finitary, note first that it holds quite generally that \( x \in \text{Cn}^y(\{x\}) \). Now suppose that \( x \in \text{Cn}^y(Y) \). There are two possibilities. One is that \( \text{Cn}^y(Y) = Y \), in which case \( x \in Y \), and so \( \{x\} \subseteq Y \). Since \( x \in \text{Cn}^y(\{x\}) \), we may conclude (by monotony (1c)) that \( x \in \text{Cn}^y(X) \) where \( X = \{x\} \) is a finite subset of \( Y \). The other possibility is that \( \text{Cn}^y(Y) = S \), in which case, even if \( \text{Cn}^y(Y) = Y \), we have \( y \in \text{Cn}(Y) \). From the finitariness (1c) of \( \text{Cn} \) it follows that \( y \in \text{Cn}(X) \), where \( X \) is a finite subset of \( Y \), which implies that \( \text{Cn}^y(X) = S \). Hence \( x \in \text{Cn}^y(X) \) where \( X \) is a finite subset of \( Y \).

**Corollary 1.** The consequence operation \( \text{Cn}^y \) is compact.

**Proof.** Suppose that \( \text{Cn}^y(Y) = S \). Then by (5), \( y \in \text{Cn}(Y) \). By (1c) applied to \( \text{Cn} \), there is a finite \( X \subseteq Y \) such that \( y \in \text{Cn}(X) \). By (5), \( \text{Cn}^y(X) = S \).

**Corollary 2.** The operation \( c^y(Y) \) defined on \( \wp(S) \) by

\[
c^y(Y) = \begin{cases} Y & \text{if } y \notin Y \\ S & \text{otherwise} \end{cases}
\]

is a compact extremal consequence operation.

**Proof.** Take for \( \text{Cn} \) in the theorem the extreme consequence operation \( \text{Cn}_{\text{min}} \).

**Lemma 2.** The following three conditions are equivalent.

(a) \( y \in \text{Cn}^y(Y) \)
(b) \( y \in \text{Cn}(Y) \)
(c) \( \text{Cn}^y(Y) = S \).

(7)
Proof. If \( y \in Cn^\eta(Y) \) and \( y \notin Cn(Y) \), then \( Cn^\eta(Y) = Y \), by (5). Hence \( y \in Y \), and so \( y \in Cn(Y) \) by (1a). It follows that if (a) \( y \in Cn^\eta(Y) \) then \( (b) \ y \in Cn(Y) \). Hence (c) \( Cn^\eta(Y) = S \), from which (a) is an immediate consequence.

**Lemma 3.** \( X \) is a \( y \)-saturated theory in \( \langle S, Cn^\eta \rangle \) if and only if it is a \( y \)-saturated theory in \( \langle S, Cn \rangle \).

**Proof.** Suppose that \( x \notin X \). It follows immediately from the equivalence of (7a) and (7b) that \( y \in Cn^\eta(X \cup \{x\}) \) if and only if \( y \in Cn(X \cup \{x\}) \), and therefore that \( X \) satisfies (2a-b) in \( \langle S, Cn^\eta \rangle \) if and only if it satisfies (2a-b) in \( \langle S, Cn \rangle \). It remains to show that \( X \) is a theory in the one calculus if and only if it is a theory in the other.

Suppose that \( X = Cn^\eta(X) \), and that \( X \) satisfies (2a-b) in \( \langle S, Cn^\eta \rangle \). Choose \( x \notin X \). We have just shown that \( X \) satisfies (2b) in \( \langle S, Cn \rangle \), and so \( y \in Cn(X \cup \{x\}) \). If \( x \in Cn(X) \) then \( y \in Cn(X) \), and so, by the equivalence of (7a) and (7b), \( y \in Cn^\eta(X) = X \). This contradicts the supposition that \( X \) satisfies (2a) in \( \langle S, Cn^\eta \rangle \). We may conclude that \( Cn(X) \subseteq X \), which means that \( X \) is a theory in \( \langle S, Cn \rangle \).

The converse is more straightforward. Assume that \( X = Cn(X) \) and that \( X \) satisfies (2a) in \( \langle S, Cn \rangle \). Then \( y \in X \), and hence \( y \notin Cn(X) \). By (5), \( Cn^\eta(X) = X \). That is, \( X \) is a theory in \( \langle S, Cn^\eta \rangle \).

**Theorem 4.** \( L(Cn^\eta, y) \) if and only if \( L(Cn, y) \).

**Proof.** Immediate.

**Theorem 5.** If the Lindenbaum extension theorem LET holds in every extremal calculus, then it holds in every calculus.

**Proof.** The result follows from Theorems 1 and 4. To allay any suspicion that there is at any point an appeal to AC, the argument is set out more explicitly below. \( C(Cn) \) means that \( Cn \) is a consequence operation and \( E(Cn) \) means that \( Cn \) is an extremal consequence operation:
Theorem 1. For all $Cn$ and all $y$: if $\mathcal{C}(Cn)$ then $\mathcal{E}(Cn^y)$

Theorem 4. For all $Cn$ and all $y$: $\mathcal{L}(Cn^y, y)$ if and only if $\mathcal{L}(Cn, y)$

Premise. For all $Cn$: if $\mathcal{E}(Cn)$ then for all $y$, $\mathcal{L}(Cn, y)$

Conclusion. For all $Cn$: if $\mathcal{C}(Cn)$ then for all $y$, $\mathcal{L}(Cn, y)$.

The argument is unquestionably valid in elementary logic (with function symbols). The Premise is Let, the restriction of LET to extremal consequence operations. The Conclusion is LET.

It was noted above that Dzik’s proof demonstrates also a result parallel to Theorem 5 for the theorem LT: if in every compact extremal calculus every consistent set has a maximal extension ($\text{Lt}$), then in every compact calculus every consistent set has a maximal extension. Using Corollary 1 to Theorem 1, we may see that the conditional $\text{Lt} \Rightarrow \text{LT}$ cannot be established, as $\text{Let} \Rightarrow \text{LET}$ was, by appeal only to the family $\mathcal{C}n^y$ of consequence operations. For the following result may be proved without appeal to AC.

Theorem 6. If $\mathcal{C}n^y$ is defined as in (5), then $\text{Lt}$ holds for the calculus $\langle S, \mathcal{C}n^y \rangle$.

Proof. If $Y \subset S$ is a consistent set in $\langle S, \mathcal{C}n^y \rangle$, then by (7), $y \notin \mathcal{C}n(Y)$. The set $S \setminus \{y\}$ is inevitably a maximal extension of $Y$. In other words, the truth of $\text{Lt}$ for all compact extremal consequence operations of the form $\mathcal{C}n^y$ does not, in the absence of AC, ensure the truth of $\text{LT}$. A proof of the conditional $\text{Lt} \Rightarrow \text{LT}$ that does not call on AC is wanting.

3. Other Results

This section records without proof a few straightforward facts about maximal and saturated theories, and some generalizations of the operations defined in (5). Lemma 7 is stated by Béziau [1] (p. 11).

Lemma 7. Let $\langle S, \mathcal{C}n \rangle$ be any calculus. The theory $Y$ is maximal in $\langle S, \mathcal{C}n \rangle$ if and only if it is $y$-saturated for every $y \notin Y$.

Lemma 8. Let $\langle S, \mathcal{C}n \rangle$ be any extremal calculus. If $Y$ is $y$-saturated in $\langle S, \mathcal{C}n \rangle$ for two distinct $y \notin Y$, then it is maximal.
Note that Lemma 8 cannot be generally improved. It is possible for $Y$ to be $y$-saturated in an extremal calculus $⟨S, Cn⟩$, but not to be maximal. For a simple example, let $S = \{a, c\}$ and $Cn(Y) = Y$ for every $Y \subseteq S$ except for $\{c\}$. Then the empty set $\emptyset$ is $a$-saturated but not maximal. On the other hand, we do have the following result.

**Lemma 9.** Let $⟨S, Cn⟩$ be any extremal calculus. If $Y$ is $y$-saturated in $⟨S, Cn⟩$, then either $Y$ is maximal or $Y \cup \{y\}$ is maximal.

We may write $Cn^{xz}$ (and similar expressions) as abbreviations for $(Cn^x)^z$ (and similar expressions). In other words

$$Cn^{xz}(Y) = \begin{cases} Y & \text{if } x \notin Cn^z(Y) \\ S & \text{otherwise.} \end{cases}$$

(8)

**Lemma 10.** $Cn^{xz}(Y) = Cn^{xz}(Y)$ for all $x, z, Y$ if and only if $Cn$ is extremal.

**Theorem 11.** $Cn^{yy}(Y) = Cn^y(Y)$ for all $y, Y$.

**Theorem 12.** $Cn^{(xz)y}(Y) = Cn^{(x)(y)}(Y)$ for all $x, y, z, Y$.

Finally we state a simple and unsurprising characterization of extremal consequence operations. For any function $ψ(Y)$, we define $C^ψ$ as follows.

$$C^ψ(Y) = \begin{cases} Y & \text{if } ψ(Y) = \emptyset \\ S & \text{otherwise.} \end{cases}$$

(9)

**Theorem 13.** The operation $Cn: \wp(S) \rightarrow \wp(S)$ is an extremal consequence operation if and only if $Cn = C^ψ$ for some function $ψ$ that satisfies (1c); that is to say

$$ψ(X) = \bigcup \{ψ(Z) \mid Z \subseteq X \text{ and } |Z| < ω₀\}.$$  

(10)

The function $Cn^y$ defined from $Cn$ in (5) is obtained by setting $ψ(Y) = Cn(Y) \cap \{y\}$. More generally, if $W$ is any set we may define $Cn^W$ by setting $ψ(Y) = Cn(Y) \cap W$: 
\( \text{Cn}^W(Y) =_{Df} \begin{cases} Y & \text{if } \text{Cn}(Y) \cap W = \emptyset \\ S & \text{otherwise.} \end{cases} \) \hspace{1cm} (11)

There are other possibilities. For example, we may define \( \psi(Y) \) to be equal to \( \emptyset \) when \( W \not\subseteq \text{Cn}(Y) \) and equal to \( W \) (or any other non-empty set) otherwise. The extremal consequence operation so generated generalizes \( \text{Cn}^\emptyset \) in another direction.

4. Further Problems

The remarks above make it clear that Dzik’s proof does not assume even the full force of Let, the Lindenbaum extension theorem restricted to extremal consequence operations. Indeed, it assumes only what we may call let: if \( \text{Cn} \) is extremal consequence operation for which

(a) there is no \( y \in S \) for which \( \text{Cn} \{y\} = S \)
(b) if \( \text{Cn}(Y) = S \) then \( \text{Cn}\{x, z\} = S \) for some \( x, z \in Y \)
(c) if \( \text{Cn}(\{x, y\}) = S = \text{Cn}(\{y, z\}) \), and \( x \neq z \),
    then \( \text{Cn}(\{x, z\}) = S \)

then there exists at least one saturated theory in \( \langle S, \text{Cn} \rangle \). We plan to discuss elsewhere whether the conditionals \( \text{let} \Rightarrow \text{LET} \) and \( \text{let} \Rightarrow \text{LT} \), or any of the weaker conditionals obtained by selecting from the list (12a-c), are demonstrable without appeal to the axiom of choice. It is a matter of some surprise that only LT, of all the equivalents of AC here considered, is recorded in the catalogue of Howard and Rubin [3].

References

