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ON THE POWER OF ULTRAFILTER LOGIC

Abstract
We examine the expressive and deductive powers of ultrafilter logic, an extension of classical first-order logic by a generalised quantifier \( \forall \), whose intended interpretation is “almost all”. We characterise a simple class of formulae from which the quantifier \( \forall \) can be eliminated, and establish that ultrafilter logic has more expressive power than classical first-order logic. We also show that, in the absence of almost universal information, the only almost universal sentences that can be derived are the universal ones.

1. Ultrafilter Logic
Ultrafilter logic extends classical first-order logic by a generalised quantifier \( \forall \), whose intended interpretation is “almost all”. In this section we briefly review ultrafilter logic: its syntax and semantics [4, 12, 14, 15, 17].

We consider a fixed signature (logical type) \( \lambda \) with a repertoire of symbols for predicates, functions and constants. We also consider a denumerably infinite set \( V \) of new symbols for variables. We let \( L[\lambda] \) be the usual first-order language (with equality \( \approx \)) of signature \( \lambda \), closed under the propositional connectives, as well as under the quantifiers \( \forall \) and \( \exists \).

We use \( L^\forall[\lambda] \) for the extension of the usual first-order language \( L[\lambda] \) obtained by adding the new operator \( \forall \).

The syntax of our logic is obtained by extending the usual first-order syntax by the new quantifier.
The formulae of $L\,\nabla[\lambda]$ are built by the usual formation rules and the following new (variable-binding) formation rule giving almost universal formulae:

$$\left(\nabla\right) \text{ for each variable } v \in V, \text{ if } \varphi \text{ is a formula in } L\,\nabla[\lambda] \text{ then so is } \nabla \varphi.$$ 

The semantics of ultrafilter logic is provided by enriching first-order structures with ultrafilters and extending the usual definition of satisfaction [5, 6] to the generalised quantifier $\nabla$.

An ultrafilter structure $M^U = (M, U)$ for signature $\lambda$ consists of a first-order structure $M$ for signature $\lambda$ together with a proper ultrafilter $U$ over the universe $M$ of $M$.

We extend the usual definition of satisfaction of a formula in a structure under an assignment to its free variables as follows

$$(|=\nabla) \text{ for a formula } \nabla \vee \psi(u, v), \text{ we define } M^U |= \nabla \nabla \vee \psi(u, v)[a] \text{ iff the extension } M^U[\psi(u, v)] \text{ belongs to the ultrafilter } U;$$

where $M^U[\psi(u, v)] := \{b \in M : M^U |= \nabla \psi(u, v)[a, b]\}$.

As usual, satisfaction of a formula depends only on the realisations assigned to its symbols. In particular, satisfaction for purely first-order formulae (without $\nabla$) does not depend on the ultrafilter, i.e. for a formula $\Theta(w)$ of $L[\lambda]$, we have $M^U |= \nabla \Theta(w)[m]$ iff $M |= \Theta(w)[m]$.

The corresponding notion of ultrafilter consequence is as expected: $\Gamma |= \nabla \forall \tau$ iff $M^U |= \nabla \forall \tau$ whenever $M^U |= \nabla \Gamma$, for every ultrafilter structure $M^U$.

The following formulae, coding properties of ultrafilters, are valid

1. $\nabla \forall \varphi \rightarrow \exists \forall \varphi$;
2. $\neg \nabla \forall \varphi \rightarrow \nabla \forall \neg \varphi$;
3. $(\nabla \forall \psi \land \nabla \forall \Theta) \rightarrow \nabla \forall (\psi \land \Theta)$.

We can set up a deductive system for our logic by adding schemata corresponding to these properties of ultrafilters to a calculus for classical first-order logic. We then obtain a sound and complete deductive calculus for ultrafilter consequence:

$$\Gamma |= \nabla \tau \text{ iff } \Gamma \vdash \nabla \tau.$$
Moreover, ultrafilter logic is a conservative extension of classical first-order logic: for $\Sigma$ and $\tau$ in $L[\lambda]$ (without $\nabla$), we have

$$\Sigma \models \nabla \tau \text{ iff } \Sigma \models \tau.$$  

We mention a few simple properties of ultrafilter logic. First, we have substitutivity of equivalents

$$\Sigma \models \nabla \psi \leftrightarrow \Theta \text{ whenever } \Sigma \models \psi \leftrightarrow \Theta.$$  

Also, we can see that, within equivalence, the new quantifier $\nabla$ – commutes with negation ($\models \nabla \neg \nabla \psi \leftrightarrow \neg \nabla \psi$), and

– distributes over the binary prepositional connectives $\land$, $\lor$, $\rightarrow$ and $\leftrightarrow$ (e.g., $\models \nabla \psi \land \Theta \leftrightarrow \nabla (\psi \land \Theta)$ and $\models \nabla (\psi \land \Theta) \leftrightarrow (\nabla \psi \land \nabla \Theta)$).

Finally, we have prenex forms, i.e. every formula $\varphi$ of $L[\nabla, \lambda]$ is equivalent to one consisting of a prefix of quantifiers ($\forall, \exists$ and $\nabla$) followed by a quantifier-free matrix $\mu$:

$$\models \nabla \varphi \leftrightarrow Q_1 v_1 \ldots Q_k v_k \mu.$$  

### 3. Elimination of $\nabla$

We shall now characterise a simple class of formulae from which the new generalised quantifier can be eliminated.

As a motivating example, imagine a consistent first-order theory $\Gamma$ expressing information concerning (flying) birds. Assume that

– we know that some birds fly (i.e. $\Gamma \vdash \exists v F(v)$),
– all birds have beaks (i.e. $\Gamma \vdash \forall v K(v)$),
– every bird is a biped (i.e. $\Gamma \vdash \forall v D(v)$),
– flying birds have wings (i.e. $\Gamma \vdash \forall v [F(v) \rightarrow W(v)]$);
– we do not know that all birds fly (i.e. $\Gamma \not\vdash \forall v F(v)$) (or, even more strongly, that we know that not all birds fly (i.e. $\Gamma \vdash \neg \forall v F(v)$), as long as $\Gamma$ is consistent.).

Now, consider the almost universal assertion

- almost all birds fly, expressed by $\nabla v F(v)$.  

Can we express the almost universal assertion $\forall vF(v)$ by an equivalent sentence without $\forall$? The question appears to have a negative answer. Moreover, the reason for this negative answer rests entirely on classical first-order reasoning, namely

$$\Gamma \not\vdash [\exists vF(v) \rightarrow \forall vF(v)].$$

The question we shall now address concerns the elimination of almost universal prefixes from purely first-order formulae. We will show why the only almost universal assertions $\forall \lor \psi$ (where $\psi$ has no $\forall$) that can be expressed without $\forall$ are the trivial ones (in the sense that $\exists v\psi \rightarrow \forall v\psi$ can be derived).

More precisely, considering a formula $\varphi$ of $L^\lambda$ and a set $\Sigma$ of sentences over signature $\lambda$ expanding signature $\lambda'(\lambda \subseteq \lambda')$, we will say that theory $\Sigma$ $\neg$-eliminates formula $\varphi$ iff there exists some formula $\Theta$ of $L^\lambda''$ (with the same free variables) for some expansion $\lambda'' \subseteq \lambda'$ of signature $\lambda' \subseteq \lambda$, such that

$$\Sigma \models (\varphi \leftrightarrow \Theta).$$

Now, given signatures $\lambda \subseteq \lambda'$, a set $\Sigma$ of sentences of $L[\lambda']$ and a formula $\psi$ in $L[\lambda]$ (so, without $\forall$), we will show that

- theory $\Sigma \subseteq L[\lambda']$ $\neg$-eliminates formula $\forall \lor \psi$ of $L^\lambda$

iff

- $\Sigma \models (\exists v\psi \rightarrow \forall v\psi)$.

Towards this goal, we consider a formula $\Theta$ of $L[\lambda'']$ (without $\forall$) where $\lambda'' \subseteq \lambda'$.

We first see, by reasoning with models, that

- if $\Sigma \models (\forall \lor \psi \rightarrow \Theta)$,
- then $\Sigma \models (\exists \lor \psi \rightarrow \Theta)$.

Indeed, otherwise, we have a $\lambda$-model $M$ of $\Sigma$ with elements $a$, such that $M \not\models \Theta[a]$ and the extension $M[\psi(a, \varphi)]$ is nonempty, whence it can be extended to a proper ultrafilter $\mathcal{U}$. This provides an ultrafilter model $M^{\mathcal{U}} = (M, \mathcal{U})$ of $\Sigma$ such that $M^{\mathcal{U}} \models (\forall \lor \psi[a])$ (and $M^{\mathcal{U}} \not\models \forall \lor \psi[a]$).

Now, the converse implication

- if $\Sigma \models (\exists \lor \psi \rightarrow \Theta)$,
- then $\Sigma \models (\forall \lor \psi \rightarrow \exists \lor \psi)$.

follows from $\models (\forall \lor \psi \rightarrow \exists \lor \psi)$. 
Next, we note, by duality, that
\[- \Sigma \models \nabla (\Theta \rightarrow \nabla \lor \psi)\]
is a necessary and sufficient condition for
\[- \Sigma \models (\Theta \rightarrow \forall \lor \psi).\]
Indeed, it suffices to apply \[\models (\neg \nabla \lor \psi \leftrightarrow \nabla \lor \neg \psi)\] to the preceding equivalence.

By combining these equivalences, we can conclude the announced necessary and sufficient condition for eliminating a \(\nabla\) prefix from first-order formula.

**Condition for eliminating \(\nabla\) prefix from purely first-order formula**

Given a formula \(\psi\) in \(L[\lambda]\) and a set \(\Sigma\) of sentences of \(L[\lambda']\) (where \(\lambda \subseteq \lambda'\)):
\[- \Sigma \models (\nabla \lor \psi \leftrightarrow \Theta),\]
for some formula \(\Theta\) in \(L[\lambda'']\), with \(\lambda'' \subseteq \lambda'\); iff
\[- \Sigma \models (\exists \lor \psi \rightarrow \forall \lor \psi).\]

4. The Power of \(\nabla\)

We shall now briefly consider the expressive power of our ultrafilter logic, showing that it is a proper extension of classical first-order logic.

We will use our previous result to show that in ultrafilter logic we have formulae that cannot be expressed within classical first-order logic, establishing that the former is a proper extension of the latter.

Now, considering a cardinal number \(\kappa\), we will call a formula \(\varphi\) of \(L[\lambda]_{\kappa}\)-eliminable iff some pure first-order theory \(\Delta \subseteq L[\lambda']\), over some expanded signature \(\lambda' \subseteq \lambda\), having models with cardinality \(\kappa\) or above can \(\nabla\)-eliminate formula \(\varphi\), i.e., for some signature \(\lambda'' \subseteq \lambda'\), we have a formula \(\Theta\) of \(L[\lambda'']\) (with the same free variables), such that
\[\Delta \models (\varphi \leftrightarrow \Theta).\]

Also, we shall call a formula \(\varphi\) of \(L[\lambda]\) logically eliminable iff the empty set \(\emptyset\) of sentences \(\nabla\)-eliminates it, i.e., for some signature \(\lambda'' \subseteq \lambda\), we have a formula \(\Theta\) of \(L[\lambda'']\) (with the same free variables), such that
\[\models (\varphi \leftrightarrow \Theta).\]

Clearly, a logically eliminable formula must be \(\kappa\)-eliminable for every \(\kappa\).
Here, we shall employ $\emptyset$ for the signature of pure equality, without any extra-logical symbols for predicates, functions or constants.

Consider the formula $\nabla \lor v \simeq u$ of $L^{\nabla}[\emptyset]$, with distinct variables $u$ and $v$.

Formula $\nabla \lor v \simeq u$ of $L^{\nabla}[\emptyset]$ is not 2-eliminable (so, not logically eliminable).

Indeed, for a pure first-order theory $\Delta \subseteq L[\lambda']$ having models with more than one element, we clearly have

$$\Delta \nmid (\exists v \simeq u \rightarrow \forall v \lor \simeq u).$$

Thus, in view of our characterisation, we can conclude that theory $\Delta$ cannot $\nabla$-eliminate the formula $\nabla \lor v \simeq u$ of $L^{\nabla}[\emptyset]$.

Also, for a signature $\mu$, with a constant symbol $c$, we have some sentences of $L^{\nabla}[\mu]$ that are not 2-eliminable. For instance, the sentences

$$\nabla \lor v \simeq c \text{ and } \nabla \lor \neq v \simeq c \text{ of } L[\mu]$$

are not 2-eliminable (whence not logically eliminable, as well).

We now consider concepts that can be expressed by sentences of ultrafilter logic, but not within classical first-order logic.

First, for distinct variables $u$ and $v$, we have in $L^{\nabla}[\emptyset]$ the sentences

$$\exists u \nabla \lor v \simeq u \text{ and } \forall u \nabla \lor \neg v \simeq u,$$

expressing, respectively, that ultrafilters are principal or non-principal.

Now, neither one of these sentences is $\aleph_0$-eliminable.

Indeed, given a pure first-order theory $\Delta \subseteq L[\lambda']$ having infinite models, assume that, for some sentence $\tau$ of $L[\lambda'']$ (with $\lambda'' \subseteq \lambda'$), we have

$$\Delta \models \nabla \forall u \nabla \lor \neg v \simeq u \leftrightarrow \tau,$$

and consider an infinite $\lambda''$-model $M$ of $\Delta$. Then, we have a non-principal ultrafilter $N$, which provides an ultrafilter model $MN = (M, N)$ of $\Delta$ such that $M^N \models \nexists v \nabla \lor \neg v \simeq u$, whence $M \models \tau$. On the other hand, the principal (ultra)filter $P$ generated by an element of $M$ provides an ultrafilter model $MP = (M, P)$ of $\Delta$ such that $MP \nmodels \forall u \nabla \lor \neg v \simeq u$, whence $M \nmodels \tau$.

Thus, we can conclude that the concepts of principal and non-principal ultrafilters can be expressed by sentences of ultrafilter logic, but not within classical first-order logic.
Expressive powers of ultrafilter and classical first-order logics

The concepts of principal and non-principal ultrafilters can be expressed
– within ultrafilter logic, respectively, by the sentences of $L^\nabla[\emptyset]$
  $\exists u \nabla \lor v \simeq u$ and $\forall u \nabla \lor \neg v \simeq u$;
– but not by sentences of $L[\lambda]$ (without $\nabla$) for any signature $\lambda$.

Another consequence of our characterisation is the result that, in the absence of almost universal information, the only almost universal consequences are the universal ones. This expresses the fact that the universe is the only set that can be guaranteed to be in every ultrafilter, a remark that can be used for a direct proof [4, 14].

More precisely, given a set $\Sigma$ of sentences and a formula $\psi$ in $L[\lambda]$ (so, without $\nabla$), we can see that

$\Sigma \models \forall v \psi$

is a necessary and sufficient condition for

$\Sigma \models^{\nabla} \nabla \lor \psi$.

Necessity is clear from $\models^{\nabla} (\forall v \psi \rightarrow \nabla \lor \psi)$ (and conservativeness).

For sufficiency, we note that

$\Sigma \models^{\nabla} \nabla \lor \psi$

yields both

$1. \ \Sigma \models^{\nabla} (\nabla \lor \psi \leftrightarrow \top)$, (where $\top$ is any valid formula of $L[\lambda]$, e.g. $v \simeq v$),

and

$2. \ \Sigma \models \exists v \psi$, from $\models^{\nabla} (\nabla \lor \psi \rightarrow \exists v \psi)$ (and conservativeness).

The former, in view of our characterisation, gives

$\Sigma \models (\exists v \psi \rightarrow \forall v \psi)$,

which, combined with the latter yields

$\Sigma \models \forall v \psi$.

Thus, in the absence of almost universal information, the almost universal consequences coincide the universal ones, giving a case where the deductive power of ultrafilter logic does not go beyond that of classical first-order logic.
Almost universal and universal consequences of pure first-order theory
For a set $\Sigma$ of sentences and a formula $\psi$ in $L[\lambda]$ (so, without $\nabla$):
$\Sigma \models \nabla \psi \lor \psi$ iff $\Sigma \models \forall \psi$.

6. Conclusion
We have examined some aspects of the expressive and deductive powers of ultrafilter logic.

We have first characterised a simple class of formulae from which the generalised quantifier $\nabla$ can be eliminated. We have then used this result to establish that ultrafilter logic has more expressive power than classical first-order logic, by showing formulae from which the new quantifier $\nabla$ cannot be eliminated. As for deductive power, we have shown that, in the absence of almost universal information, the only almost universal sentences that can be derived are the universal ones.

Our ultrafilter logic is a proper conservative extension of classical first-order logic with compactness and Lowenheim-Skolem properties. The apparent conflict with Lindstrom’s results [9] is explained because we are using a non-standard notion of model (due to the ultrafilters).

As a logic with generalised quantifiers, ultrafilter logic is connected to such extension of first-order logic [1, 8]. It is also related to nonmonotonic reasoning [2, 10], which was one of its motivations [3, 11]. This logic also appears to have some interesting connections with inductive and empirical reasoning [7] as well as with fuzzy logic [13], suggesting the possibility of other applications for it [4, 14, 17].

References
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