ON SOME MISCONCEPTIONS ABOUT ULTRAFILTER LOGIC

Abstract
We examine some apparent misconceptions about ultrafilter logic, an extension of classical first-order logic by a generalized quantifier \( \nabla \), whose intended interpretation is “almost all”. These arise from some alleged objections against the usage of ultrafilters for capturing the intended meaning of “almost universal” assertions. The two alleged objections on which we focus concern degree of arbitrariness and trivialization in finite situations. We argue that these two apparent objections can be overcome by a proper understanding of the roles of models and theories.

1. Introduction
In this note we examine some apparent misconceptions about ultrafilter logic. Ultrafilter logic is an extension of classical first-order logic by a generalized quantifier \( \nabla \), whose intended interpretation is “almost all”.

Ultrafilter logic intends to capture directly the intuition of a property holding for a large set of elements [3, 4] and to serve as a precise basis for generic reasoning [4, 17, 19]. For this purpose, one extends (conservatively) classical first-order logic by adding a new generalized quantifier \( \nabla \) with intended interpretation “almost all” [4, 13, 17, 19].

As a logic with generalized quantifiers, ultrafilter logic is connected to such extensions of first-order logic [1, 11]. It is also related, though to a lesser extent, to the tradition of analysis and formalization of language (as
in [6, 9, 14], for instance), as well as to “common-sense” reasoning (as one of the original motivations for extending first-order logic was capturing such reasoning [12], and providing an alternative to non-monotonic logic [3]). Logics such as ultrafilter logic appear to have some interesting applications [4, 16, 17]. Probable candidates for applications of such logics are the realm of imprecise reasoning, in the spirit of fuzzy logic [15] and the area of inductive reasoning, as in empirical experiments and tests [10].

The meaning of “almost all objects have a given property” can be given either directly or by means of the set of exceptions, i.e. those objects failing to have this property. The idea is that “almost all objects have a given property” is intended to mean that the set of exceptions is “negligible”, or the set of objects having this property is “almost as important as” the entire universe [18].

The semantics of ultrafilter logic is given by means of ultrafilter structures which expand first-order structures by ultrafilters on their universes. The semantical interpretation of $\forall \lor \varphi$ is to be “$\varphi$ is almost universally valid”, i.e. the set of elements satisfying $\varphi$ is large in the sense of belonging to the given ultrafilter.

In this note we consider some apparent objections to ultrafilter logic. These alleged objections concern the usage of ultrafilters for capturing the intended meaning of “almost universal” assertions. The two alleged objections we shall address here are as follows.

1. Ultrafilters involve a certain degree of arbitrariness.
2. Ultrafilters are not interesting for finite situations.

We shall argue that these two objections can be overcome by a proper understanding of the roles of models and theories.

One may also cast some doubt on the assertion that ultrafilters provide a natural way to capture the idea of “almost all”. This possible objection can be circumvented by means of an analysis of the underlying ideas [18].

To make the note self contained we first briefly review ultrafilter logic.

2. Ultrafilter Logic

Ultrafilter logic extends classical first-order logic by a generalized quantifier $\forall$, whose intended interpretation is “almost all”. In this section we briefly review ultrafilter logic: its syntax and semantics [4, 13, 17, 19].
We consider a fixed signature (logical type) $\lambda$ with a repertoire of symbols for predicates, functions and constants. We also consider a denumerably infinite set $V$ of new symbols for variables. We let $L[\lambda]$ be the usual first-order language (with equality $=$) of signature $\lambda$, closed under the propositional connectives, as well as under the quantifiers $\forall$ and $\exists$.

We use $L^\triangledown[\lambda]$ for the extension of the usual first-order language $L[\lambda]$ obtained by adding the new operator $\triangledown$.

The syntax of our logic is obtained by extending the usual first-order syntax by the new quantifier.

The formulae of $L^\triangledown[\lambda]$ are built by the usual formation rules and the following new (variable-binding) formation rule giving almost universal formulae:

$$(\triangledown) \text{ for each variable } v \in V, \text{ if } \varphi \text{ is a formula in } L^\triangledown[\lambda] \text{ then so is } \triangledown \varphi.$$  

The semantics of ultrafilter logic is provided by enriching first-order structures with ultrafilters and extending the usual definition of satisfaction [5, 7] to the generalized quantifier $\triangledown$.

An ultrafilter structure $M^U = (M, \mathcal{U})$ for signature $\lambda$ consists of a first-order structure $M$ for signature $\lambda$ together with a proper ultrafilter $\mathcal{U}$ over the universe $M$ of $M$.

We extend the usual definition of satisfaction of a formula in a structure under an assignment to its free variables as follows

$$(\models \triangledown) \text{ for a formula } \triangledown y \psi(x, y), \text{ we define } M^U \models \triangledown y \psi(x, y)[\bar{a}] \text{ iff the extension } M^U[\psi(\bar{a}, y)] \text{ belongs to the ultrafilter } \mathcal{U};$$

where $M^U[\psi(\bar{a}, y)] := \{b \in M : M^U \models \triangledown \psi(x, y)[\bar{a}, b]\}$.

As usual, satisfaction of a formula depends only on the realizations assigned to its extra-logical symbols. In particular, satisfaction for purely first-order formulae (without $\triangledown$) does not depend on the ultrafilter, i.e. for a formula $\theta(y)$ of $L[\lambda]$, we have $M^U \models \triangledown \theta(\bar{y})[\bar{m}]$ iff $M \models \theta(\bar{y})[\bar{m}]$.

The corresponding notion of ultrafilter consequence is as expected: $\Gamma \models \triangledown \tau$ iff $M^U \models \triangledown \tau$ whenever $M^U \models \triangledown \Gamma$, for every ultrafilter structure $M^U$.

The following formulae, coding properties of ultrafilters, are valid

$$\triangledown \lor \varphi \rightarrow \exists \lor \varphi, \quad \neg \triangledown \lor \psi \rightarrow \triangledown \lor \neg \varphi, \quad (\triangledown \lor \psi \lor \triangledown \lor \theta) \rightarrow \triangledown \lor (\psi \lor \theta), \quad \triangledown \lor (\psi \lor \theta) \rightarrow (\triangledown \lor \psi \lor \triangledown \lor \theta).$$
We thus have substitutivity of equivalents:

\[ \Sigma \models \nabla \psi \leftrightarrow \nabla \theta \] whenever \( \Sigma \models \psi \leftrightarrow \theta \).

Also, we can see that, within equivalence, the new quantifier \( \nabla \) – commutes with negation (\( \models \nabla \neg \psi \leftrightarrow \nabla \neg \phi \)), and

– distributes over the binary propositional connectives \( \land, \lor, \to \), and \( \leftrightarrow \) (e.g., \( \models \nabla \psi \land \theta \leftrightarrow \nabla \psi \land \nabla \theta \)) and \( \models \nabla \psi \lor \phi \leftrightarrow (\nabla \psi \lor \nabla \phi \lor \theta) \).

We can set up a deductive system for our logic by adding schemata coding properties of ultrafilters to a calculus for classical first-order logic. We then obtain a sound and complete deductive calculus for ultrafilter consequence:

\[ \Gamma \models \tau \text{ iff } \Gamma \vdash \tau. \]

Moreover, ultrafilter logic is a conservative extension of classical first-order logic: for \( \Sigma \) and \( \tau \) in \( L[\lambda] \) (without \( \nabla \)), we have:

\[ \Sigma \models \tau \text{ iff } \Sigma \models \tau. \]

3. Arbitrariness

We shall now examine the contention that an ultrafilter embodies a certain degree of arbitrariness. The idea is that, even though this may be the case for a given ultrafilter, this is largely dissolved when one considers, and reasons with, theories.

As an example, let us consider the set \( \mathbb{N} \) of the natural numbers and the two (infinite) subsets of \( \mathbb{N} \):

– the set \( E \) of the even numbers, and

– its complement \( E^c \) (the set of odd numbers).

These two infinite subsets have the same cardinality and appear to be equally important, but exactly one of them must be negligible. Now, considering either one of them as negligible, and the other one as very important, appears to be somewhat arbitrary.

Sometimes, a context (given by an application) may remove this feeling of arbitrariness (rendering it only apparent). For instance, the set \( P \) of prime numbers may be rightfully deemed very important by a number-theorist working on problems of Cryptography.

Now, let us return to the general case, where ultrafilters may appear to involve some arbitrariness.
First, we recall an example of a proper filter over an infinite universe $V$. It is the so-called Fréchet filter consisting of the cofinite subsets of $V$ (i.e. those with finite complement) [2, 5].

Now, consider the Fréchet filter $\mathcal{F}$ over the universe of $\mathbb{N}$ of the naturals. Given any infinite subset $Z$ of $\mathbb{N}$, we can see that the family $\mathcal{F} \cup \{Z\}$ has the finite intersection property (for any finite family of sets $X_1, \ldots, X_n$ in $\mathcal{F} \cup \{Z\}$, we have $X_1 \cap \cdots \cap X_n \neq \emptyset$, otherwise $Z$ would be finite). Hence, it can be extended to a proper ultrafilter $U_Z$ over $\mathbb{N}$, i.e. $\mathcal{F} \cup \{Z\} \subseteq U_Z$.

So, we have several proper ultrafilters over $\mathbb{N}$, for instance

- an ultrafilter $U_E$, with $E \in U_E$ (hence $E^c \notin U_E$), and
- an ultrafilter $U_{E^c}$, with $E^c \in U_{E^c}$ (hence $E \notin U_{E^c}$).

The set $E$ of the even numbers is very important in the former case, but not in the latter, when the set $E^c$ of the odd numbers is very important.

This situation is depicted in Figure 1 below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{filters}
\caption{Filters over the naturals}
\end{figure}
We thus have ultrafilter models where the assertion “almost all numbers are even” holds, as well as models where it fails to hold. Hence, in a theory having such models the assertion “almost all numbers are even” is left undecided: neither it nor its negation is provable.

In this sense, the alleged degree of arbitrariness, that may be attached to a given model, is largely dissolved when one considers theories.

This fact is corroborated by a result of ultrafilter logic: in the absence of almost universal information, the only almost universal consequences are the universal ones [4, 16]. This expresses the fact that the universe is the only set that can be guaranteed to be in every ultrafilter.

A perhaps pertinent analogy is with probability. Probability theory is more concerned with obtaining some probabilities from others, by means of properties, than with assigning probabilities to particular events. Particular distributions may assign different probabilities to evens and odds, but, without information about distributions, one would know very little about their probabilities.

Even in a theory with almost universal information expressing that the cofinite sets are very important (i.e. a finite set has almost no number), which can be expressed by the assertion $\forall y \forall x \neg x = y$ (see next section), the assertion “almost all numbers are even” is still left undecided.

4. Finiteness

We shall now examine the contention that ultrafilters are not interesting for finite situations. Again, the idea is that, even though this may be the case for a particular (finite) structure, this is practically dissolved when one considers theories.

Clearly, over a finite universe $V$, every ultrafilter is finite. Moreover, as it is well known, such an ultrafilter must be generated by a singleton (since a finite universe has a finite cover by all its singletons). So, such an ultrafilter consists of all the subsets including some element of $V$, i.e. it must be of the form 

$$g\mathcal{U} = \{X \subseteq V : g \in X\}$$

for some element $g \in V$ (its generator) [2, 5].

For ultrafilter $g\mathcal{U}$ generated by $g \in V$, we have:

- almost all objects have a property ($\varphi(v)$, i.e. $\forall \wedge (\varphi(v)$) iff

  - the generator $g$ has this property, i.e. $\varphi(g)$. 

This equivalence provides a crucial test for property $\varphi(v)$, reducing almost all to element $g$. So, such an element $g$ may be termed archetypal or generic for this property [4, 16]. (The term “generic” has been used in other contexts, such as [8], for a similar, but not quite the same, idea.)

One must, however, bear in mind that we are making these considerations in the presence of a given ultrafilter. It is in this context that

- the almost universal assertion $\nabla \lor \varphi(v)$

reduces to

- the simpler assertion $\varphi(g)$.

In a given finite ultrafilter model, we have its ultrafilter, so we can assume to know its generator.

Let us now consider several finite ultrafilter structures, each one with its own ultrafilter. We know that each such ultrafilter has a generator, but we may not have access to it.

A similar situation occurs with theories. Even though we may know that each ultrafilter has a generator, we may be unable to identify it.

For instance, imagine a (consistent) theory $\Gamma$ expressing information concerning flying birds, such as

- almost all birds fly (i.e. $\Gamma \vdash \nabla \lor F(v)$),
- Mother Goose does fly (i.e. $\Gamma \vdash F(m)$),
- Woody, a woodpecker, flies (i.e. $\Gamma \vdash F(p)$),
- Sam, a penguin, does not fly (i.e. $\Gamma \vdash \neg F(s)$).

Assuming that the universe $B$ of birds is finite, we can be sure that each ultrafilter will have a generator, i.e. we know that

$$B^\mathcal{U} \models \exists y \forall x x = y.$$  

But, we do not know its identity. In fact, we have very little information concerning the generator. Besides the fact that it cannot be the non-flying Sam, for all we know the generator might be

- either Woody,
- or Mother Goose,
- or even some unnamed bird.

Thus, if Tweety is (the name of) a bird, the theory will not decide whether it flies or not.

Having names for all the possible objects in a finite universe will not change the situation. For, even though the theory guarantees the existence of an object (a generator), it still may happen that all we can know is that
it is one of the objects in the finite universe, without being able to decide
the disjunction and identify it.

For a universe with three objects, consider theory $\Delta$ involving three
constants, say $s$ (for “solid”), $l$ (for “liquid”), and $k$ (for “gaseous”), with
axioms stating that these three constants are pairwise distinct and exhaust
the entire universe (e.g. $\neg s = l$, $\neg s = k$, $\neg l = k$, and $\forall x[x = s \lor x = l \lor x = k]$). We will then have

$$\Delta \vdash \exists y \forall x x = y,$$

whence, also

$$\Delta \vdash \forall x (x = s \lor x = l \lor x = k);$$

but, theory $\Delta$ will not decide this disjunction, which is to be expected
(two are no grounds to deem an element much more important than the
others).

Indeed, over the universe $\{s, l, k\}$, we have three ultrafilters, namely
- ultrafilters $\mathcal{U}_s$, $\mathcal{U}_l$, and $\mathcal{U}_k$,
with respective generators $s, l$, and $k$.

(These three ultrafilters are displayed in Figure 2 below).

So, theory $\Delta$ has three ultrafilter models
- $\ast C, l C$, and $k C$,
expanding the three-element universe $\{s, l, k\}$, respectively, with the ultra-
filters $\mathcal{U}_s$, $\mathcal{U}_l$, and $\mathcal{U}_k$.

Each of these models decides about the disjunction in favor of the
generator of its ultrafilter, e.g. $\ast C \models \forall x x = s$, but $\ast C \models \forall x x = l$ and
$\ast C \models \forall x x = k$.

In this sense, the contention that ultrafilters over a finite universe
are not interesting (because then they are generated by an element) is
dissolved when one considers theories. Even though in each particular
finite structure, its ultrafilter has a generating element, this does not mean
that a theory will be able to pinpoint such a generator (for all its models).
5. Conclusion

We have examined some apparent misconceptions about ultrafilter logic.

Ultrafilter logic intends to capture directly the intuition of a property holding for a large set of elements and to serve as a precise basis for generic reasoning. For this purpose, one extends (conservatively) classical first-order logic by adding a new generalized quantifier \( \nabla \) with intended interpretation “almost all” \([4, 13, 17, 19]\).

These alleged objections concern the usage of ultrafilters for capturing the intended meaning of “almost universal” assertions. The two alleged objections in connection with ultrafilters we have addressed here are:
1. degree of arbitrariness, and
2. trivialization in finite situations.

We have argued that these two objections can be overcome by a proper understanding of the roles of models and theories.

Another apparent objection (lack of intuitive justification for using ultrafilters) can be circumvented by reassessing the ultrafilter interpretation in terms of local and relative notions concerning sets (the relation of “having about the same importance”, as well as the properties of being “negligible” and “very important”). An analysis of our intuitive understanding of these basic notions suggests some reasonable postulates, from which we can rigorously derive the characteristic properties of ultrafilters [18].

We have begun by examining the first alleged objection (arbitrariness in connection with ultrafilters). Indeed, this may be the case with a particular ultrafilter (since it must have either a set or else its complement). We have, however, argued that this is largely dissolved when one considers theories (with models for both alternatives).

Next, we have taken up the second alleged objection to ultrafilters (trivialization in finite situations). Indeed, this may be so with a particular ultrafilter (since, being over a finite universe, it must have a generator, providing a crucial test for the almost universal assertions). Again, this objection is dissolved when one considers theories (with several ultrafilter models). Then, the mere existence of a generator (established by the theory) does not identify it (and the crucial test is lost).

In conclusion, we have examined two alleged objections against the usage of ultrafilters for capturing the intended meaning of “almost universal” assertions. We trust that our analysis of the contentions concerning (apparent) degree of arbitrariness and trivialization in finite situations has helped to dispel some possible misconceptions about ultrafilter logic. Logics such as this one appear to merit further investigation, both as logical systems and in connection with other fields, which may suggest interesting applications or variants of these ideas.
References


Praça Eugênio Jardim, 6/apt. 501;
22061-040 Rio de Janeiro RJ; BRASIL