EXHAUSTIVELY AXIOMATIZING $RMO\rightarrow$ WITH A SELECT LIST OF REPRESENTATIVE THESSES INCLUDING RESTRICTED MINGLE PRINCIPLES

Abstract

$RMO\rightarrow$ is the result of adding the ‘mingle principle’ [i.e., $A \rightarrow (A \rightarrow A)$] to Anderson’s and Belnap’s implicative logic of relevance $R\rightarrow$. The aim of this paper is to provide all possible axiomatizations with independent axioms of $RMO\rightarrow$ formable with Anderson and Belnap’s “strong and natural list of valid entailments” extended with ten characteristic minglish principles.

Keywords: exhaustive axiomatization, relevance logics, non-classical logics.

1. Introduction

Let $R\rightarrow$ be the implication fragment of Anderson and Belnap’s Logic of Relevance $R$. Then, $RMO\rightarrow$ is the result of adding the axiom $A \rightarrow (A \rightarrow A)$ (‘mingle’) to $R\rightarrow$. [See [1], §8.15]. Méndez has shown in [4] how to extend $RMO\rightarrow$, with conjunction, disjunction and negation in order to obtain and define the Logics of Relevance $RMOm$ and $RMO$. These logics are suggested as possible alternatives to Anderson and Belnap’s $R$; therefore, $RMO\rightarrow$ has to be understood as an alternative to the pure implicative logic of relevance $R\rightarrow$. 

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This paper is a sequel to [2], [3] and [5]. [2] shows how to “exhaustively” axiomatize $R_\rightarrow$ with Anderson and Belnap’s “strong and natural list of valid entailments” ([1], p.26). [5] exhaustively axiomatizes $RMO_\rightarrow$ with an extension of this list with three characteristic minglish principles. Finally, in [3] it is shown how to exhaustively axiomatize the implicative fragments of Lewis S3 and S4 relative to an extension of Anderson’s and Belnap’s list with some characteristic modal theorems. The aim of this paper is to exhaustively axiomatize $RMO_\rightarrow$ with an extension of Anderson and Belnap’s list with ten characteristic minglish principles. These include restricted mingle theorems such as those treated in [6].

Exhaustive axiomatizations in general, and those here presented in particular, provide proof-theoretical results whose proofs are effective and independent of the choice of semantics. For example, exhaustive axiomatization provides conservative (un)extendability results for implicative logics. These results allow to represent logics with a given property absent in the embedding ones. For example, the systems with the Ackermann Property or with the Decidability Property are representable in logics without them. Hence, one of the points of these exhaustive axiomatizations is the definition of extensions of $R$ with the forementioned properties.

2. Extending Anderson and Belnap’s list

Anderson and Belnap’s list is the following [see [1], §8.15]:

1. $A \rightarrow A$
2. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$
3. $(B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
4. $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$
5. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
6. $(A \rightarrow B) \rightarrow [[(A \rightarrow (B \rightarrow C))] \rightarrow (A \rightarrow (D \rightarrow C))]$
7. $(D \rightarrow B) \rightarrow [(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow (D \rightarrow C))]$
8. $(C \rightarrow D) \rightarrow [(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow (B \rightarrow D))]$
9. $[A \rightarrow (B \rightarrow C)] \rightarrow [A \rightarrow [(D \rightarrow B) \rightarrow (D \rightarrow C)]]$
10. $[A \rightarrow (B \rightarrow C)] \rightarrow [A \rightarrow [(C \rightarrow D) \rightarrow (B \rightarrow D)]]$
11. $[A \rightarrow [(B \rightarrow C) \rightarrow D]] \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow D)]$
12. $(B \rightarrow C) \rightarrow [[[A \rightarrow [(B \rightarrow C) \rightarrow D]]] \rightarrow (A \rightarrow D)]$
13. $(A \rightarrow B) \rightarrow [[(A \rightarrow B) \rightarrow C] \rightarrow C$
14. $[(A \rightarrow A) \rightarrow B] \rightarrow B$
For $R\rightarrow$ we add (see [2]):

15. $[A \rightarrow (B \rightarrow C)] \rightarrow [B \rightarrow (A \rightarrow C)]$
16. $B \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (A \rightarrow C)]$
17. $A \rightarrow [(A \rightarrow B) \rightarrow B]$

and for $RMO\rightarrow$,

18. $A \rightarrow (A \rightarrow A)$
19. $(A \rightarrow B) \rightarrow [A \rightarrow (A \rightarrow B)]$
20. $(A \rightarrow B) \rightarrow [B \rightarrow (A \rightarrow B)]$
21. $(A \rightarrow B) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow B)]$
22. $[(A \rightarrow B) \rightarrow C] \rightarrow [(A \rightarrow B) \rightarrow [(A \rightarrow B) \rightarrow C]]$
23. $[(A \rightarrow (B \rightarrow C)] \rightarrow [A \rightarrow (A \rightarrow (B \rightarrow C))]]$
24. $[(A \rightarrow B) \rightarrow (A \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]]$
25. $[(A \rightarrow B) \rightarrow C] \rightarrow [C \rightarrow [(A \rightarrow B) \rightarrow C]]$
26. $[A \rightarrow (B \rightarrow C)] \rightarrow [(B \rightarrow C) \rightarrow [A \rightarrow (B \rightarrow C)]]$
27. $[(A \rightarrow B) \rightarrow (C \rightarrow D)] \rightarrow [(C \rightarrow D) \rightarrow [(A \rightarrow B) \rightarrow (C \rightarrow D)]]$

We note that 21 is a restricted version of 19 [it is, in fact, restricted mingle] and 22, 23 and 24 (25, 26 and 27) restricted versions of 19 (20). These theses [21-27] are the restricted mingle versions that are treated in [6].

3. Syntactic lemmas

In the proofs to follow, we shall use the following conventions:

(i) The conditional associates to the left

(ii) The dot separates the main antecedent and consequent of an implicational wff

(iii) The dot also separates the main antecedent and consequent of a subordinate wff if it is within brackets

For example,

$(D \rightarrow B) \rightarrow [(A \rightarrow (B \rightarrow C)] \rightarrow [A \rightarrow (D \rightarrow C)]$

Is rendered by,

$(D \rightarrow B) \rightarrow A \rightarrow (B \rightarrow C) \rightarrow [A \rightarrow .D \rightarrow C]$

or by,

$(D \rightarrow B) \rightarrow A \rightarrow (B \rightarrow C) \rightarrow [A \rightarrow (D \rightarrow C)]$
Additionally, by “trans” we shall refer to the rule: if \( \varphi A \rightarrow B \) and \( \varphi B \rightarrow C \), then \( \varphi A \rightarrow C \).

**Lemma 1.**

(a) 17 is derivable from 1 and 15  
(a’) 14 is derivable from 1 and 16  
(a”’) 17 is derivable from 1, 16 and 3  
(b) 3 is derivable from 14 and 2 (7,8,9,10)  
(c) 4 is derivable from 3, 14 and 5 (6)  
(d) 1 is derivable from 3 and 14.

In (a)–(d), the numerals refer to the theses in the list. The sole rule of inference is modus ponens (MP).

**Note.** In Lemma 1 of [2] items (a), (a’) and (a”’) are replaced by (a*): 17 is derivable from 1 and 15 (16). However, none of the conclusions reached in [2] are threatened, as it is subsidiarily shown below.

Now we prove Lemma 1. Cases (a) and (d) are immediate. Case (c)(i): 4 is derivable from 3, 14 and 5. It is then immediate. Case (c)(ii): 4 is derivable from 3, 14 and 6. Immediate once we note [case d] that 1 is derivable. Case (a’): 14 is derivable from 1 and 16:

1. \( A \rightarrow A \rightarrow A \rightarrow A \rightarrow B \rightarrow [A \rightarrow A \rightarrow B] \rightarrow [A \rightarrow A \rightarrow B \rightarrow B] \) \[16\]  
   By 1 and MP.

2. \( A \rightarrow A \rightarrow B \rightarrow B \)

Case (a’’): 17 is derivable from 1, 16 and 3:

1. \( A \rightarrow B \rightarrow (A \rightarrow B) \rightarrow [A \rightarrow B \rightarrow B] \rightarrow [A \rightarrow B \rightarrow B] \) \[14\]  
   By 1, 2 and 3.

2. \( A \rightarrow A \rightarrow B \rightarrow (A \rightarrow B) \rightarrow [A \rightarrow B \rightarrow B] \) \[16\]  
   By 1, 2 and 3.

3. \( A \rightarrow A \rightarrow B \rightarrow B \)

**Note.** In the preceding proof, the rule trans would be sufficient.

Case (b)(i): 3 is derivable from 14 and 2. First we prove that from 14 and 2 the rule R:

If \( \varphi A \rightarrow B \), then \( \varphi A \rightarrow B \rightarrow C \rightarrow C \) is derivable. Suppose
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1. $A \rightarrow B$
   Then,
2. $A \rightarrow B \rightarrow .B \rightarrow B \rightarrow (A \rightarrow B)$ [2]
3. $B \rightarrow B \rightarrow .A \rightarrow B$ [MP 1,2]
   By 3 and 2,
4. $A \rightarrow B \rightarrow C \rightarrow [B \rightarrow B \rightarrow C]$ By 4 and 2,
5. $B \rightarrow B \rightarrow C \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$
Thus, by 14,
6. $A \rightarrow B \rightarrow C \rightarrow C$

Now, 3 is derivable as follows:

1. $B \rightarrow C \rightarrow .C \rightarrow C \rightarrow (B \rightarrow C)$
2. $C \rightarrow C \rightarrow (B \rightarrow C) \rightarrow .B \rightarrow C \rightarrow (A \rightarrow C) \rightarrow$
   $[C \rightarrow C \rightarrow (A \rightarrow C)]$ [2]
   By 1, 2 and 2,
3. $B \rightarrow C \rightarrow .B \rightarrow C \rightarrow (A \rightarrow C) \rightarrow [C \rightarrow C \rightarrow (A \rightarrow C)]$
4. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow (A \rightarrow C)$
   By 4 and 2,
5. $B \rightarrow C \rightarrow (A \rightarrow C) \rightarrow [C \rightarrow C \rightarrow (A \rightarrow C)] \rightarrow .A \rightarrow B \rightarrow$
   $[C \rightarrow C \rightarrow (A \rightarrow C)]$
   By 3, 5 and 2,
6. $B \rightarrow C \rightarrow .A \rightarrow B \rightarrow [C \rightarrow C \rightarrow (A \rightarrow C)]$
7. $A \rightarrow B \rightarrow [C \rightarrow C \rightarrow (A \rightarrow C)] \rightarrow .C \rightarrow C \rightarrow (A \rightarrow C) \rightarrow$
   $(A \rightarrow C) \rightarrow [A \rightarrow B \rightarrow (A \rightarrow C)]$
   By 6, 7 and 2,
8. $B \rightarrow C \rightarrow .C \rightarrow C \rightarrow (A \rightarrow C) \rightarrow (A \rightarrow C) \rightarrow [A \rightarrow B \rightarrow$
   $(A \rightarrow C)]$ By 14 and the rule R,
9. $C \rightarrow C \rightarrow (A \rightarrow C) \rightarrow (A \rightarrow C) \rightarrow [A \rightarrow B \rightarrow (A \rightarrow C)] \rightarrow$
   $[A \rightarrow B \rightarrow (A \rightarrow C)]$
   Finally, by 8, 9 and 2,
10. $B \rightarrow C \rightarrow .A \rightarrow B \rightarrow (A \rightarrow C)$

Now we prove case (b)(ii). 3 is derivable from 14 and 7. First we prove that the rule $\text{trans}$ is derivable from 14 and 7. Suppose
1. $A \rightarrow B$
   And
2. $B \rightarrow C$
Now,
3. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)]$  \hspace{1cm} [7]
4. $B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)]$  \hspace{1cm} [MP 1,3]
5. $B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)] \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)]$ \hspace{1cm} [14]

By 2, 4, 5 and MP,
6. $A \rightarrow C$

Now we have:
1. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)]$  \hspace{1cm} [7]
2. $B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)] \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)]$ \hspace{1cm} [14]

But line 3 is 2, and, then, 3 is derivable as in case (b)(i). Finally, we note that the other subcases of (b) are proved similarly.

**Lemma 2.** (Cfr. Lemma 2 of [2]) $R_\rightarrow$ may be axiomatized (with Modus Ponens) using any selection that includes one (and only one) thesis from the following groups:

$\{1, 14\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}$

**Proof** of Lemma 2 is as follows. Given that $R_\rightarrow$ is axiomatized with 1, 3, 4 and 17 (see [1]), Lemma 1(a) shows that 17 can be replaced by 15 in this axiomatization. Lemma 1(b), that 3 can be replaced by 2, 7, 8, 9 or 10 in each one of the preceding axiomatizations [note that 14 follows from 1 and 17]. Lemma 1(a") shows that 15 (17) can be replaced by 16 in each of the preceding axiomatizations [note that 14 and 1 are derivable - by Lemmas 1(a"), 1(d)- and that by Lemma 1(b), 3 is always present]. Lemma 1(c) shows that 4 is interchangeable with 5 or 6 in each of the eighteen previous axiomatizations. Finally, Lemmas 1(b), 1(d) show that 14 can replace 1 in each of the fifty four previous axiomatizations of $R_\rightarrow$.

**Lemma 3.**

(a) 18 is derivable from 1 and 19 (20)
(b) 21 is derivable from 1 and 27 (26, 25, 24, 23, 22)
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(c) 18 is derivable from 1, 2, 15 and 21
(d) 19-27 are derivable from 1, 2, 15 and 18

Before proving Lemma 3, we note that we immediately have:

(i) 2 (3) is derivable from 3 (2) and 15
(ii) 17 is derivable from 1 and 15
(iii) $A \rightarrow A \rightarrow A \rightarrow A$ is derivable from 1 and 17.

Now, (a) and (b) are immediate, and (d) can be proved as follows. First we prove that 19 is derivable from 1, 2, 15 and 18:

1. $A \rightarrow A \rightarrow A \rightarrow B \rightarrow (A \rightarrow B)$  \[2\]
   By 1, 2 and 18,
2. $A \rightarrow .A \rightarrow B \rightarrow (A \rightarrow B)$
   By 2 and 15,
3. $A \rightarrow B \rightarrow .A \rightarrow (A \rightarrow B)$

20 is similarly proved, and 22-27 are immediate being as they are restricted forms of 19 and 20. Finally, 21 is a restricted form of 18.

The proof of Lemma 3 ends with the proof of case (c):

1. $A \rightarrow A \rightarrow A \rightarrow .A \rightarrow A \rightarrow A \rightarrow [A \rightarrow A \rightarrow A]$ \[21\]
   As 17 is derivable from 1 and 15 [cfr. (ii) above],
2. $A \rightarrow .A \rightarrow A \rightarrow A$
   By 1, 2 and 2,
3. $A \rightarrow .A \rightarrow A \rightarrow A \rightarrow [A \rightarrow A \rightarrow A]$
4. $A \rightarrow [A \rightarrow A \rightarrow A] \rightarrow .A \rightarrow A \rightarrow A \rightarrow [A \rightarrow A \rightarrow A] \rightarrow [A \rightarrow .A \rightarrow A]$ \[2\]
   By 3, 5 and 2,
5. $A \rightarrow A \rightarrow A \rightarrow [A \rightarrow A \rightarrow A] \rightarrow .A \rightarrow [A \rightarrow A \rightarrow A]$ \[MP 2,4\]
   By 3, 5 and 2,
6. $A \rightarrow .A \rightarrow [A \rightarrow A \rightarrow A]$
   Since 3 is derivable [Cfr. (i) above],
7. $A \rightarrow A \rightarrow A \rightarrow A \rightarrow .A \rightarrow [A \rightarrow A \rightarrow A] \rightarrow (A \rightarrow A)$ \[3\]
   By $A \rightarrow A \rightarrow A \rightarrow A$ [Cfr. (iii) above] and MP,
8. $A \rightarrow [A \rightarrow A \rightarrow A] \rightarrow (A \rightarrow A)$
   Finally, by 6, 8 and 2,
9. $A \rightarrow .A \rightarrow A$

This ends the proof of Lemma 3, from which we immediately have:
Lemma 4. 18-27 are equivalent whenever 1, 2 and 15 are present.

Lemma 5. The rule \texttt{trans} is derivable from 18 and 2 (3,7,8,9,10)

Proof: It is obvious for 2 and 3. Let us prove:

\texttt{Trans} follows from 7 and 18 (other cases are similar).

So, suppose:

1. \( A \rightarrow B \)
2. \( B \rightarrow C \)
3. \( A \rightarrow B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)] \) \[7\]
4. \( B \rightarrow C \rightarrow (B \rightarrow C) \rightarrow [B \rightarrow C \rightarrow (A \rightarrow C)] \) \[MP 1,3\]
5. \( B \rightarrow C \rightarrow .B \rightarrow C \rightarrow (B \rightarrow C) \) \[18\]
6. \( B \rightarrow C \rightarrow .B \rightarrow C \) \[MP 2,5\]
7. \( A \rightarrow C \) \[MP 2,4,6\]

With Lemma 5 we prove:

Lemma 6.

(a) 1 is derivable from 4 and 18
(b) 1 is derivable from 5, 18 and 17 (15)
(c) 1 is derivable from 5, 18, 16 and \texttt{trans}
(d) 1 is derivable from 6, 18 and 17 (15)
(e) 1 is derivable from 6, 18, 16 and \texttt{trans}.

Note. The Lemma stated in [5] replaces (b) and (c) by:

(iv) 1 is derivable from 5, 18 and 17 (15,16) and replaces (d) and (e) by:

(v) 1 is derivable from 6, 18 and 17 (15,16)

as it happened with Lemma 1, these replacements are seemingly wrong. Nevertheless we note that, again, all conclusions in the paper [5] in fact still hold, as it is also shown below.

Proof of Lemma 6. Since case (a) is immediate, we prove cases (b) and (c), proofs of cases (d) and (e) being, respectively, similar.

Case (b).

(i) 1 is derivable from 5, 18 and 17

From 5 and 17, 4 is immediate. Hence, 1 follows as in case (a).

(ii) 1 is derivable from 5, 18 and 15
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By 5 and 18,
1. $A \rightarrow A \rightarrow .A \rightarrow A$
   By 15,
2. $A \rightarrow .A \rightarrow A \rightarrow A$
   By 2 and 5,
3. $A \rightarrow (A \rightarrow A) \rightarrow .A \rightarrow A$

Now, 1 follows by 18 just as in case (a).

Case (c). 1 is derivable from 5, 18, 16 and trans.

By 5 and 18,
1. $A \rightarrow A \rightarrow .A \rightarrow A$
   2. $A \rightarrow A \rightarrow .A \rightarrow A \rightarrow (A \rightarrow A)$
      [18]
   By 2, 18 and trans.
3. $A \rightarrow .A \rightarrow A \rightarrow (A \rightarrow A)$
   4. $A \rightarrow .A \rightarrow A \rightarrow (A \rightarrow A) \rightarrow [A \rightarrow A \rightarrow A]$    [16]
   By 4 and 5,
5. $A \rightarrow [A \rightarrow A \rightarrow (A \rightarrow A)] \rightarrow .A \rightarrow [A \rightarrow A \rightarrow A]$
   6. $A \rightarrow .A \rightarrow A \rightarrow A$ [MP 3,5]

Now 1 is derivable as in (ii) of case b.

From Lemmas 3-6 we have Lemmas 7,8:

LEMMA 7. $RMO_{\rightarrow}$ may be axiomatized (with Modus Ponens) using any selection that includes one (and only one) thesis of the following groups:
   \{1, 14\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}, \{18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}

Proof of Lemma 7 is as follows. By Lemma 1, each one of the selections formulable with the first four groups is an axiomatization of $R_{\rightarrow}$. Now, given that $RMO_{\rightarrow}$ is $R_{\rightarrow}$ plus 18, Lemma 4 shows that 18 can be replaced by any of 19-27.

LEMMA 8. $RMO_{\rightarrow}$ may be axiomatized (with Modus Ponens) using any selection that includes one (and only one) thesis from the following groups:
   \{18\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}

Proof of Lemma 8 is obvious given Lemmas 5 and 6, since they show that 1 is derivable from any of these 54 formulations. Therefore, each one of them is an axiomatization of $RMO_{\rightarrow}$, as inspection of the groups in Lemma 7 shows.
### 4. Matrices

We provide five matrices to be used in the independence proofs of section 5. Designated values are starred.

**MATRIX I**

<table>
<thead>
<tr>
<th>→ 0 1 2</th>
<th>Verifies</th>
<th>Falsifies</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 2 2 2</td>
<td>2-13</td>
<td>1 (A = 1)</td>
</tr>
<tr>
<td>1 0 0 2</td>
<td>15-17</td>
<td>14 (A = B = 1)</td>
</tr>
<tr>
<td>*2 0 0 2</td>
<td>19-27</td>
<td>18 (A = 1)</td>
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</table>

**MATRIX II**

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<th>Falsifies</th>
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</thead>
<tbody>
<tr>
<td>0 2 2 2</td>
<td>1-14</td>
<td>15 (A = 2, B = C = 1)</td>
</tr>
<tr>
<td>1 0 2 2</td>
<td>18-27</td>
<td>16 (A = 2, B = C = 1)</td>
</tr>
<tr>
<td>*2 0 0 2</td>
<td></td>
<td>17 (A = B = 1)</td>
</tr>
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**MATRIX III**

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<th>Falsifies</th>
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</thead>
<tbody>
<tr>
<td>0 2 2 2</td>
<td>1-3</td>
<td>4 (A = 1, B = 0)</td>
</tr>
<tr>
<td>1 1 2 2</td>
<td>6-27</td>
<td>5 (A = B = 1, C = 0)</td>
</tr>
<tr>
<td>*2 0 1 2</td>
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<td>6 (A = B = 1, C = 0)</td>
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**MATRIX IV**

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<tbody>
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<td>0 3 3 3 3</td>
<td>1-17</td>
<td>18 (A = 1)</td>
</tr>
<tr>
<td>1 0 2 0 3</td>
<td></td>
<td>19 (A = B = 1)</td>
</tr>
<tr>
<td>*2 0 1 2 3</td>
<td></td>
<td>20 (A = B = 1)</td>
</tr>
<tr>
<td>*3 0 0 0 3</td>
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<td></td>
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</table>

**MATRIX V**

<table>
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<th>→ 0 1 2 3 4</th>
<th>Verifies</th>
<th>Falsifies</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 3 3 3 3 3</td>
<td>1,4-6</td>
<td>2,3 (A = 3, B = 2, C = 1)</td>
</tr>
<tr>
<td>1 0 4 2 3 4</td>
<td>11-17</td>
<td>7 (A = 1, B = 2, C = 1, D = 3)</td>
</tr>
<tr>
<td>*2 0 1 2 3 4</td>
<td>18-27</td>
<td>8 (A = 3, B = 1, C = 2, D = 1)</td>
</tr>
<tr>
<td>*3 0 0 2 3 0</td>
<td></td>
<td>9 (A = 1, B = 2, C = 1, D = 3)</td>
</tr>
<tr>
<td>*4 0 1 2 3 4</td>
<td></td>
<td>10 (A = 3, B = 1, C = 2, D = 1)</td>
</tr>
</tbody>
</table>
Exhaustively axiomatizing $RMO_\rightarrow$

**Note 1.** Matrix IV falsifies 21-27: conditionals can take any of the values 0,1,2,3 and 21-27 are restricted versions of the falsified theses 18-20.

**Note 2.** Matrices I-V show that at least one thesis in the groups

\{1, 14, 18\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}, \{18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}

has to be added for axiomatizing $RMO_\rightarrow$. Lemma 8 implies that if 18 is chosen in the first group, 19-27 are derivable and $RMO_\rightarrow$ is axiomatized no matter which theses is taken from any of the other groups. Lemma 7 shows that if 1 or 14 are chosen, $RMO_\rightarrow$ is axiomatized no matter which theses are selected from any other groups.

5. Exhaustively axiomatizing $RMO_\rightarrow$

Finally, we prove:

**Theorem** $RMO_\rightarrow$ may be axiomatized (with Modus Ponens) using any selection that includes one (and only one) thesis from each of the groups in (a) and (b) below:

(a) \{1, 14\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}, \{18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}

(b) \{18\}, \{2, 3, 7, 8, 9, 10\}, \{4, 5, 6\}, \{15, 16, 17\}

The 1,134 resulting selections (the order of the axioms is not taken into account) are the only axiomatizations of $RMO_\rightarrow$ with independent axioms formulable with the twenty seven thesis in the list.

To prove that all selections in (a) and (b) have independent axioms, use the matrices in section 4 as follows:

- independence of 1, 14 and 18: Matrix I
- independence of 2, 3, 7, 8, 9 and 10: Matrix V
- independence of 4, 5 and 6: Matrix III
- independence of 15, 16 and 17: Matrix II
- independence of 18, 19, 20, 21, 22, 23, 24, 25, 26 and 27: Matrix IV

We put an end to the proof noting that given Lemmas 7 and 8, it suffices to inspect the matrices in §4 to see that all axiomatizations (with independent axioms) of $RMO_\rightarrow$, formulable with 1-27 are exactly those of (a) and (b) in the above theorem [note that 11, 12 and 13, being restricted versions of 15, 16 and 17, have not been employed]
References


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