ON THE EQUIVALENCE BETWEEN TWO SYSTEMS OF PARACONSISTENT LOGIC

The system $P_1$ was introduced by A. M. Sette in [6]. It is a system of maximal paraconsistent logic which is currently the focus of much research (see [1], [4] and [5]).

The postulates of $P_1$ are the following:
1) $A \supset (B \supset A)$
2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3) $(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A)$
4) $\neg(A \supset \neg\neg A) \supset A$
5) $(A \supset B) \supset \neg\neg(A \supset B)$
6) $A, A \supset B/B$

(In [1], it is shown that axiom 4 is not independent.)

$P_1$ is complete relative to the following matrix, where 1 and 2 are the designated values (see [6], p. 177): $M = \langle 1, 2, 3; \supset, \neg \rangle$. The connectives $\supset$ and $\neg$ are defined by the tables:

<table>
<thead>
<tr>
<th>$\supset$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\neg A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

The other connectives are introduced by contextual definitions.

In [3] a different system called $F$ is studied with the following axioms:
1) $A \supset (B \supset A)$
2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3) $A \supset (B \supset A \& B)$
4) $A \& B \supset A$
5) $A \& B \supset B$
6) $(A \supset C) \supset ((B \supset C) \supset (A \lor B \supset C))$
7) \( A \supset A \lor B \)
8) \( B \supset A \lor B \)
9) \( A, A \supset B/B \)
10) \( \neg\neg A \supset A \)
11) \( A \lor \neg A \)
12) \( \neg(A \lor \neg A) \) if \( A \) is not atomic.

In [3] the authors claimed that system \( F \) coincides with Sette’s system \( P_1 \) (see [3], p. 85). However, this is not true. To show this it is enough to verify that the formula \( \neg(B \land \neg B) \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)) \), which is a theorem of \( P_1 \), is not valid in \( F \). To see this we can use the following matrix \( M = <\{1, 2, 3\}, \{1, 2, 3\}, \supset, \land, \neg, >: \)

<table>
<thead>
<tr>
<th>( \supset )</th>
<th>1 2 3</th>
<th>&amp;</th>
<th>1 2 3</th>
<th>( \neg A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 1 3</td>
<td>1 1 3</td>
<td>1 3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 1 3</td>
<td>2 1 3</td>
<td>2 2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 1 1</td>
<td>3 3 3</td>
<td>3 1</td>
<td></td>
</tr>
</tbody>
</table>

We can correct the mistake by adding the formula \( \neg(B \land \neg B) \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)) \) as a new axiom of \( F \). With this modification, the completeness theorem \( (\Gamma \models A \Rightarrow \Gamma \vdash A) \) becomes valid (because the strong negation of the system: \( \neg^*A = \neg A \land \neg (A \land \neg A) \) now has all the properties of classical negation). This fact was, erroneously, claimed before (see [3], p. 85).

As we have a new axiom, we also have to modify the definition of \textit{valuation} proposed for the system \( F \). The original definition was as follows:

“A valuation of \( F \) is a function \( \lor \) from the set of formulas of \( F \) to the set \( \{0, 1\} \), satisfying the following conditions:

1) For \( \land, \supset \) and \( \lor \), the same rules as for the classical propositional calculus;
2) \( \lor(A) = 0 \Rightarrow \lor(\neg A) = 1, \) if \( A \) is an atomic formula;
3) \( \lor(A) = 0 \Leftarrow \lor(\neg A) = 1, \) if \( A \) is not an atomic formula.”

(see [3], p. 84).

Now, we have to add the rule:

4) \( \lor(\neg(B \lor \neg B)) = \lor(A \supset B) = \lor(A \supset \neg B) = \lor(A) = 1 \Rightarrow \lor(A) = 0. \)

With this new rule, the following theorem is valid:

\textbf{Soundness theorem.} \( \Gamma \vdash A \Rightarrow \Gamma \models A. \)
With the above corrections, the system $F$ becomes equivalent to the system $P_1$ and, as such, it is an extension of system $C_1$ of da Costa. As a matter of fact, as it was defined before, $F$ is an extension of system $C_\omega$, also of da Costa (see [2] for an overview of systems $C_1$ and $C_\omega$).

References