Judit Madarász

THE CRAIG INTERPOLATION THEOREM IN MULTI-MODAL LOGICS

In [13] and [14] Maksimova proved that a normal modal logic (with one unary modality) has the Craig interpolation property iff the corresponding class of algebras has the superamalgamation property. (These notions will be recalled below.) In this paper we extend Maksimova’s theorem to normal multi-modal logics with arbitrarily many, not necessarily unary modalities, and to not necessarily normal multi-modal logics with modalities of ranks smaller than 2.

Before formulating our theorems, let us review some definitions. Given a modal similarity type (i.e., a set of modalities and a function assigning a finite rank to each modality), the set of modal formulas is built up from the propositional variables, the constant 1 (True) and the constant (i.e. zero-ary) modalities by means of the Boolean connectives ¬, ∨ and the modalities the usual way. In particular, if ♦ is an n-ary modality, and ϕ₁, ϕ₂, . . . , ϕₙ are formulas then ♦(ϕ₁, ϕ₂, . . . , ϕₙ) is a formula, too. We use ♦ to denote modalities. For each n-ary modality ♦, the dual □ of ♦ is defined as follows: □(ϕ₁, ϕ₂, . . . , ϕₙ) = ¬♦(¬ϕ₁, ¬ϕ₂, . . . , ¬ϕₙ). Here ϕ₁, ϕ₂, . . . , ϕₙ are formula variables. We denote propositional variables by pᵢ (i is a natural number).

The definition of a multi-modal logic as given in our Definition 1 below is already enough for our present purposes, but for completeness we would like to note that it has a disadvantage. Namely, identifying a logic with its validities unnecessarily narrows down the application areas of logic. Therefore, in Remark 1 below, we give a more general definition, too. Our Definition 1 is given in the spirit of those in [13] p. 458 or in [19] p. 151.

Definition 1. (multi-modal logic, normal multi-modal logic, in the narrow sense) We define a multi-modal logic to be a set L of multi-modal formulas such that

(i) L contains the classical tautologies, and Axiom (K) for every modality in each of its arguments, where (K) is recalled below.
(K) \( \Box(p_1, \ldots, p_{i-1}, p \rightarrow p', p_{i+1}, \ldots, p_n) \rightarrow \)
\( (\Box(p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n) \rightarrow \Box(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n)) \),

where \( \Box \) is an n-ary modality, and \( 1 \leq i \leq n \).

(ii) \( L \) is closed under modus ponens, substitution and conditional necessitation (CN), recalled below.

(CN) If \( \phi \rightarrow \phi' \in L, \Box \) is n-ary and \( 1 \leq i \leq n \) then
\( \Box(\phi_1, \ldots, \phi_{i-1}, \phi, \phi_{i+1}, \ldots, \phi_n) \rightarrow \Box(\phi_1, \ldots, \phi_{i-1}, \phi', \phi_{i+1}, \ldots, \phi_n) \in L. \)

A normal multi-modal logic is a multi-modal logic \( L \) which is closed under necessitation (N), recalled below.

(N) If \( \phi \in L, \Box \) is an n-ary and \( 1 \leq i \leq n \) then
\( \Box(\phi_1, \ldots, \phi_{i-1}, \phi, \phi_{i+1}, \ldots, \phi_n) \in L. \)

Convention 1. If \( L \) is a logic then \( Fm_L \) denotes the set of all formulas of \( L \).


In a setting more general than our Definition 1, a logic \( L \) is identified with a consequence relation \( \vdash_L \) between \( \mathcal{P}(Fm_L) \) and \( Fm_L \) where \( Fm_L \) is the set of all formulas of \( L \). In more detail, \( \vdash_L \subseteq \mathcal{P}(Fm_L) \times Fm_L \) satisfying the following condition (i) due to Tarski.

(i) For any \( \Gamma, \Delta \subseteq Fm_L \) and \( \phi \in Fm_L, \)
\( \phi \in \Gamma \implies \Gamma \vdash_L \phi; \) and
\[ [\Gamma \vdash_L \phi, \text{ and for every } \psi \in \Gamma \text{ we have } \Delta \vdash_L \psi] \implies \Delta \vdash_L \phi. \]

In this spirit, we call \( L \) a multi-modal logic iff \( Fm_L \) is as defined above Definition 1, all validities of classical propositional logic as well as Axiom (K) are derivable from the empty set by \( \vdash_L \), and \( \vdash_L \) is closed under modus ponens \( \phi, (\phi \rightarrow \psi) \vdash_L \psi, \) substitution, and the rule of conditional necessitation (CN), i.e.,
\( \phi \rightarrow \psi \vdash_L \)
\( \Box(\phi_1, \ldots, \phi_{i-1}, \phi, \phi_{i+1}, \ldots, \phi_n) \rightarrow \Box(\phi_1, \ldots, \phi_{i-1}, \psi, \phi_{i+1}, \ldots, \phi_n) \)
for every \( \phi, \psi, \phi_1, \ldots, \phi_n \in Fm_L, \) and \( 1 \leq i \leq n. \)
We note that compact multi-modal logics are algebraizable in the sense of [5], [6], and equivalently, they are strongly nice in the sense of [1], [17]. A difference between [5], [6] and [1], [17] is that [5], [6] require (algebraizable) logics to be compact while [1], [17] do not require (strongly nice) logics to be such. We do not assume compactness, either. In the rest of this paper we will ignore this difference.

For any strongly nice or algebraizable logic \( L \) there is a natural class \( \text{Alg}(L) \) of algebras associated with \( L \) in e.g. [17], [1], [5] and [6]. \( \text{Alg}(L) \) consists of the natural formula-algebras (often called Lindenbaum-Tarski algebras) associated with the theories of \( L \), and all isomorphic copies of these. (The class \( \text{Alg}(L) \) is called 'equivalent algebraic semantics' in [5], and 'equivalent quasivariety' in [6].) \( 
V(\text{Alg}(L)) \) denotes the variety (equational class) generated by \( \text{Alg}(L) \).

Only the definition of \( 
V(\text{Alg}(L)) \) has to be recalled in detail for the purposes of the present paper. Next we turn to doing that. With any multi-modal logic \( L \), we can associate an algebraic similarity type as follows. If \( f \) is a logical connective of \( L \) (i.e., \( f \) is either a Boolean connective or a modality) then \( f \) has a finite rank \( n \). On the algebra side, we can consider \( f \) to be a function symbol of arity \( n \). Let \( t \) denote the similarity type consisting of the logical connectives of \( L \), considered as function symbols, with arities as we defined above. E.g., if \( \circ \) is an \( n \)-ary modality of \( L \) then \( \circ \) will be an \( n \)-ary algebraic operation of \( \text{Alg}(L) \). Then formulas of \( L \) can be thought of as terms in the algebraic language of similarity type \( t \). (On the algebra side, we interpret the propositional variables \( p_i \) as algebraic variables.) Now \( 
V(\text{Alg}(L)) \) is the class of all algebras \( A \) of type \( t \), such that the identity \( \phi = \psi \) is valid in \( A \) for every \((\phi \to \psi) \in L \).

**Definition 2.** Let \( L \) be a multi-modal logic. We say that \( L \) has the \( \to \text{Craig} \) interpolation property (in the terminology of Maksimova [13]) \( L \) has CIP if for any formulas \( \phi \) and \( \psi \), the condition \((\phi \to \psi) \in L \) implies that there exists a formula \( \chi \) such that \((\phi \to \chi) \in L \), \((\chi \to \psi) \in L \), and \( \text{Voc}(\chi) \subseteq \text{Voc}(\phi) \cap \text{Voc}(\psi) \), where \( \text{Voc}(\gamma) \) denotes the set of propositional variables occurring in \( \gamma \), for any formula \( \gamma \).

**Definition 3.** (cf. Maksimova [13], [14]) By a partially ordered algebra we mean a structure \((A, \leq)\) where \( A \) is an algebra and \( \leq \) is a partial ordering on the universe \( A \) of \( A \).

A class \( K \) of partially ordered algebras has the superamalgamation
property (SUPAP) if for any \( A_0, A_1, A_2 \in K \) and for any embeddings \( i_1 : A_0 \rightarrow A_1 \) and \( i_2 : A_0 \rightarrow A_2 \) there exist an \( A \in K \) and embeddings \( \epsilon_1 : A_1 \rightarrow A \) and \( \epsilon_2 : A_2 \rightarrow A \) such that \( \epsilon_1 \circ i_1 = \epsilon_2 \circ i_2 \) and

\[
(\forall x \in A_j)(\forall y \in A_k)(\epsilon_j(x) \leq \epsilon_k(y) \Rightarrow (\exists z \in A_0)(x \leq i_j(z) \land i_k(z) \leq y)),
\]

where \( \{j, k\} = \{1, 2\} \).

We note that Maksimova writes ‘SAP’ for (SUPAP). (We write (SUPAP) because a large part of the literature uses ‘SAP’ to abbreviate ‘Strong Amalgamation Property’ cf. [15], [2], [3], [11].) If \( L \) is a multi-modal logic and \( A \in V(Alg(L)) \), then \( A \) is partially ordered by the relation \( \leq \) defined as follows: \( x \leq y \) iff \( \neg x \lor y = 1 \).

**Theorem 1.** Let \( L \) be a normal multi-modal logic (possibly with modalities of ranks greater than 1). Then

\( L \) has the ‘\( \rightarrow \) Craig’ interpolation property \( \iff V(Alg(L)) \) has (SUPAP).

**Theorem 2.** Let \( L \) be a multi-modal logic with all modalities of ranks smaller than 2. (Note that \( L \) need not be normal.) Then

\( L \) has the ‘\( \rightarrow \) Craig’ interpolation property \( \iff V(Alg(L)) \) has (SUPAP).

The class \( BAO \) of Boolean Algebras with Operators and the concept of normal Boolean Algebras with Operators were introduced in [8], cf. also in [9], [16], [10] and [18].

There is a very strong connection between modal logics and \( BAO \)'s. Namely, let \( L \) be a (not necessarily normal) multi-modal logic. Then \( V(Alg(L)) \subseteq BAO \). In the other direction, every \( BAO \) variety coincides with \( V(Alg(L)) \) for some multi-modal logic \( L \).

Assume that \( L \) is either an algebraizable logic in the sense of [5], [6] or equivalently, it is a strongly nice general logic in the sense of [1], [17]. If \( Alg(L) \subseteq BAO \) then the usual propositional connectives, e.g. ‘\( \rightarrow \)’, are available in \( L \). Therefore it is meaningful to speak about the ‘\( \rightarrow \)Craig’ interpolation property.

In the following theorem we use the terminology of [5], [6] or [1], [17] (it is enough to be familiar with only one of these).
We recall from [3] that a BAO $A$ is weakly normal if for every operator $f(x_1, \ldots, x_n)$ of $A$, the derived operation $(f(x_1, \ldots, x_n) - f(0, \ldots, 0))$ is normal (in $A$).

**Theorem 3.** Assume that $L$ is either an algebraizable logic in the sense of [5], [6] or equivalently, it is a strongly nice general logic in the sense of [1], [17]. Assume that $Alg(L)$ consists of weakly normal BAO’s. Then

$L$ has the ‘→ Craig’ interpolation property $\iff V(Alg(L))$ has (SUPAP).

To put Theorems 1, 2 in perspective, we recall the following theorem from [3].

**Theorem 4.** (Thm. 7 in [3])

(i) There is a multi-modal logic $L$ with one binary modality $\Diamond$, such that $L$ is not equivalent with any normal multi-modal logic, i.e. for no normal multi-modal logic $L_1$, $V(Alg(L))$ is term definitionally equivalent with $V(Alg(L_1))$.

(ii) There is a variety $V$ of BAO’s such that $V$ is not term definitionally equivalent to any variety of normal BAO’s.

**Definition 4.** Let $L$ be a (not necessarily modal) logic in the general sense of [1] or [17] or [5] or [6], and ‘→’ a binary connective of $L$. We say that $L$ has the Weak ‘→ Craig’ interpolation property iff the following condition holds:

For any $\phi, \psi \in \text{Fm}_L$, whenever $\vdash_L \phi \rightarrow \psi$ and $\{ \chi \in \text{Fm}_L : \text{Voc}(\chi) \subseteq \text{Voc}(\phi) \cap \text{Voc}(\psi) \} \neq \emptyset$ hold then also $\vdash_L \phi \rightarrow \chi$ and $\vdash_L \chi \rightarrow \psi$ hold for some $\chi \in \text{Fm}_L$ with $\text{Voc}(\chi) \subseteq \text{Voc}(\phi) \cap \text{Voc}(\psi)$.

**Remark 2.** The Weak ‘→ Craig’ interpolation property is equivalent with the ‘→ Craig’ interpolation property in logics containing the constant True. Thus in multi-modal logics these two Craig interpolation properties are equivalent.

Below we will show that for direction ‘$\iff$’ of Theorem 3 above we do not have to restrict ourselves to modal logics or equivalently to BAO’s. We will see that ‘$\iff$’ is true for practically all logics which are called algebraizable in [5] and [6] or strongly nice in [1] and [17]. However in order to talk about something that is called ‘→ Craig’ interpolation property we
need to assume the existence of a binary connective say \( \vee \) of our logic \( L \) which we will treat as the connective ‘\( \rightarrow \)’ in the definition of ‘\( \rightarrow \) Craig’ interpolation property. It is useful to keep in mind that connective \( \vee \) does not have to coincide with the usual Boolean ‘\( \rightarrow \)’. In Theorem 5 we will define (from connective ‘\( \rightarrow \)’) a binary relation ‘\( \leq \)’ on our algebras. The definition will be unambiguous, because \( L \) is assumed to be algebraizable (strongly nice). (Below we will not assume that ‘\( \leq \)’ would be a partial ordering. This causes no problem because in the definition of (SUPAP) we did not use that ‘\( \leq \)’ is a partial ordering.)

**Theorem 5.** Let \( L \) be an algebraizable logic in the sense of [5] and [6] or let \( L \) be strongly nice in the sense of [1] and [17]. Assume \( L \) has a binary connective which we denote by ‘\( \rightarrow \)’. Let \( \leq \) be the natural binary relation associated with ‘\( \rightarrow \)’ on the algebras of \( L \); that is for any theory \( T \) of \( L \), in the Lindenbaum-Tarski algebra \( \mathcal{Fm}_L/T \) we define

\[
\phi/T \leq \psi/T \iff T \vdash_L (\phi \rightarrow \psi),
\]

where \( \mathcal{Fm}_L \) is the formula algebra of \( L \), and \( \gamma/T \) is the equivalence class of formula \( \gamma \) (and so \( \gamma/T \) is an element of \( \mathcal{Fm}_L/T \)). In \( \text{Alg}(L) \) we understand (SUPAP) w.r.t. the relation \( \leq \) associated with ‘\( \rightarrow \)’. Then

\begin{align*}
( \text{\( L \) has the Weak ‘\( \rightarrow \) Craig’ interpolation property} ) & \iff \text{\( \text{Alg}(L) \) has (SUPAP)} .
\end{align*}

Direction ‘\( \Rightarrow \)’ fails for some logic \( L \) satisfying the conditions of Theorem 5 (cf. e.g. [2]).

**Proposition 1.** Let \( t \) be a similarity type for some BAO’s. Then the class \( \text{BAO}_t \) of all (not necessarily normal) BAO’s of similarity type \( t \) has (SUPAP). Therefore, the corresponding general multi-modal logic has the ‘\( \rightarrow \) Craig’ interpolation property.

**Theorem 6.** Assume the conditions of Theorem 5. Assume that \( \text{Alg}(L) \) has at least one constant symbol. Then,

\begin{align*}
\text{L has ‘\( \rightarrow \) Craig’ interpolation property} & \iff \text{\( V(\text{Alg}(L)) \) has (Free } \leq \text{ SUPAP) defined below.}
\end{align*}

Let \( V \) be a variety with a distinguished binary relation \( \leq \) defined on its elements. \( \mathcal{F}(X) \) denotes the free \( V \) algebra generated by set \( X \). Then

\[\text{152}\]
\( V \) is said to have \((\text{Free} \leq \text{SUPAP})\) iff for any \(X, Y\) with \(F(X \cap Y) \neq O\) letting \(i_X : F(X \cap Y) \rightarrow F(X), i_Y : F(X \cap Y) \rightarrow F(Y)\) to be the natural (identity) injections, the diagram consisting of \(i_X\) and \(i_Y\) can be superamalgamated in \(V\) as described in Definition 3.

Acknowledgements. This research was supported by Hungarian National Foundation for Scientific Research grant (OTKA) No’s T016448, T7255, T7567 and F17452. The author would like to thank István Németi and to Ildikó Sain for encouragement, to András Simon, to Ágnes Kurucz and to the other members of the Budapest Algebraic Logic Group for their help.

References


c/o Ildikó Sain
Mathematical Institute of the Hungarian Academy of Science
Budapest, Pf.127, H-1364, Hungary
e-mail: mjutka@ludens.elte.hu

154