In the preceding issue of this Bulletin (vol. 19, no 3, p. 108) Professor Wolniewicz posed the following question about join-semilattices $L$ with unit:

“Under what properties of $L$ – short of all its maximal ideal being finite – does an antichain $\text{MinV}(A)$ always contain a minimal subchain $B$ such that $r(B) = r(\text{MinV}(A))$?”

The very formulation of the question indicates that Wolniewicz knows the following answer which we write down as Theorem 1.

**Theorem 1.** Let $L$ be a non-degenerate join-semilattice with unit. Assume that each maximal ideal of $L$ is finite. Then every antichain of the form $M = \text{MinV}(A)$ contains a minimal subset $B \subseteq M$ such that $r(B) = r(M)$.

1. In this section we make comment on Wolniewicz’s answer. We first note that Theorem 1 can be stated in a bit more general form:

**Theorem 1’**. Let $L$ be a join-semilattice with unit. Assume that each maximal ideal of $L$ is finite. Then for every $M \subseteq L$ there is $B \subseteq M$ such that $r(B) = r(M)$ and $B$ is minimal with respect to this property.

**Proof.** Define $\mathcal{B} = \{B \subseteq M : r(B) = r(M)\}$. (Recall that $r(M) = \{R : M \cap R \neq \emptyset \text{ and } R \text{ is a maximal ideal of } L\}$.) We check that $\mathcal{B}$ is closed under arbitrary intersections of chains and the partial order $(\mathcal{B}, \supseteq)$ fulfills the assumption of Kuratowski-Zorn’s Lemma. Then a maximal element in $(\mathcal{B}, \supseteq)$ is a minimal one in $\mathcal{B}$. □

To grasp the idea of Theorem 1 we formulate now several equivalents of the assumption to the effect that every maximal ideal is finite.
Proposition 1. Let $L$ be a join-semilattice with unit. Then the following four conditions are equivalent:

(i) Every proper ideal of $L$ is finite;
(ii) Every maximal ideal of $L$ is finite;
(iii) Every proper ideal of $L$ is principal and finite;
(iv) $L$ satisfies ACC (the ascending chain condition) and every proper and principal ideal of $L$ is finite.

Notice that $L$ satisfies ACC iff every (nonempty) ideal of $L$ is principal. Moreover, none of the conditions (i)-(iv) implies the finiteness of $L$ as shown by the semilattice represented by the following diagram:

![Diagram showing a semilattice with conditions (i), (ii), (iii), and (iv).](image)

2. At this point we wish to discuss the notion of minimality in $B$-like families of sets. We get rid of algebraic assumptions, and therefore, we let $L$ be an arbitrary non-empty set and $\mathcal{R}$ be a non-empty family of non-empty subsets of $L$. Then we define for $M \subseteq L$

$$r(M) = \{ R \in \mathcal{R} : R \cap M \neq \emptyset \}$$

and

$$\mathcal{B} = \{ B \subseteq M : r(B) = r(M) \}.$$ 

Proposition 2. Under the above assumptions the following conditions are equivalent:
(a) There is a minimal element in \((\mathcal{B}, \subseteq)\);

(b) There is a \(B_0 \in \mathcal{B}\) such that:

\[ (*) \quad (\forall b \in B_0)(\exists R \in R)(R \cap B_0 = \{b\}). \]

**Proof.** (a) \(\rightarrow\) (b). Let \(B_0\) be a minimal element in \((\mathcal{B}, \subseteq)\). Then \(r(B_0) = r(M)\) and \(r(B') \neq r(B_0)\) for all \(B' \subset B_0\) such that \(B' \neq B_0\). In particular, for every \(b \in B_0\), \(r(B_0 - \{b\}) \neq r(B_0)\). Hence there is \(R \in R\) such that \(R \cap B_0 \neq \emptyset\) and \(R \cap (B_0 - \{b\}) = \emptyset\). So \(R \cap B_0 = \{b\}\) which proves that our \(B_0\) satisfies condition \((*)\).

(b) \(\rightarrow\) (a). Let \(B_0 \in \mathcal{B}\) fulfill \((*)\). Suppose that \(B_0\) is not minimal in \((\mathcal{B}, \supseteq)\). Hence there is \(B' \subset B_0\) such that \(r(B') = r(M)\) and \(B' \neq B_0\). Take \(b \in B_0\) such that \(b \not\in B'\). Since \(B' \subset B_0 - \{b\} \subset B_0\) we have \(r(B') \subset r(B_0 - \{b\}) \subset r(B_0)\) which contradicts the hypothesis that \(B_0\) fulfills \((*)\). As a result \(B_0\) is minimal. \(\square\)

Proposition 2 furnishes an answer to Wolniewicz’s question when we take \(R\) to be a set of all maximal ideals in a non-degenerate join-semilattice \(L\) with unit. Another answer is given by the following corollary:

**Corollary.** Let \(M \subset L\), and let \(1 \not\in M\) where \(1\) is the unit of \(L\). Assume that a set \(B_0 \in \mathcal{B}\) satisfies condition:

\[(**) \quad (\forall b, b' \in B_0)(b \neq b' \rightarrow b \lor b' = 1). \]

Then \(B_0\) is minimal in \(\mathcal{B}\).

**Proof.** We shall prove that our set \(B_0\) satisfies the separation condition \((*)\) of Proposition 2. Let then \(b \in B_0\), and let \((b)\) be the principal ideal generated by \(b\). Since \(1 \not\in M\), \((b)\) is proper. Hence there is a maximal ideal \(R\) which extends \((b)\). We show that \(R \cap B_0 = \{b\}\). Suppose then that there is \(b' \neq b\) such that \(b' \in R \cap B_0\). So \(b \lor b' \in R\). But by \((**), b \lor b' = 1\), which means that \(R\) is improper. This contradiction proves that our \(B_0\) satisfies condition \((*)\) of Proposition 2, and thus \(B_0\) is minimal.
3. At this point we furnish two examples. Firstly, we give a counterexample to Wolniewicz’s parenthetic remark: “Actually we have in general: if $B \subset A$ and $r(A) = r(B)$ then $r(A - B) = r(B)$.”

Let $L$ be a join-semilattice represented by the diagram below, and let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.

We have $r(A) = r(B)$, but $r(A - B) \neq r(B)$ since $(a) \notin r(A - B)$.

Secondly, we give an example of a join-semilattice $F$ such that the family $B = \{B \subset M : r(B) = r(M)\}$, where $M$ is the set of all minimal elements of $F$, has no minimal subset. The example is due to T. Furmanowski who has kindly agreed to state it here.

Furmanowski’s semilattice $F$ is represented by the following diagram.
It should be easily seen that the universe of $F$ is identical with the set
$((\omega \cup \{\omega\}) \times \omega) \cup \{\omega + 1\}$ and

(1) $R$ is a maximal ideal of $F$ iff $R$ is a principal ideal generated by an element of the form $(\omega, n)$ for $n = 0, 1, 2, \ldots$

(2) $M = \{(0, n) : n \in \omega\}$ is the set of all minimal elements of $F$ and, moreover, $r(M)$ = the set of all maximal ideals of $F$;

(3) For every $B \subset M$, $r(B) = r(M)$ iff $B$ is infinite.

The above conditions imply that the family $\mathcal{B}$ has no minimal element.

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