Bulletin of the **S**ection of **L**ogic
Volume 19/3 (1990), pp. 74–78

reedition 2005 [original edition, pp. 74–78]

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FINITE LOGICS AND THE SIMPLE SUBSTITUTION PROPERTY

Abstract

In [6] is defined the simple substitution property and in [4], [6] and [7] are given some examples of the sets of axioms which have the property. In this paper, we prove that the set of axioms for finite logics provided in [2] to prove that the finite logics are axiomatizable, has the property.

We abbreviate the simple substitution property as SSP.

1 Preliminaries

For the intuitionistic propositional logic, we use the system LJ presented by Gentzen. We use, possibly with suffixes, lower case Latin letters for propositional variables, and upper case Latin letters for formulas. For logical connectives, we use the symbols $\neg, \wedge, \vee$ and $\supset$ as usual, and we assume the order of priority of this order. Formulas are also defined as usual. And further, we assume the association to the left concerning each one of the logical connectives. In the following, we write $A \equiv B$ for $(A \supset B) \wedge (B \supset A)$.

If a logic L is obtained from LJ by adding new axioms $A_1, A_2, \ldots, A_n$, we write L as

$$LJ + A_1 + A_2 + \ldots + A_n$$

and we regard the set of additional axioms $\{A_1, A_2, \ldots, A_n\}$ as the set of axioms for L.

The operations $\cup$ (union) and $\cap$ (intersection) on logics are defined as usual (see eg. [3]).

For a formula $B, \Pi_n(B)$ is the conjunction of all the formulas obtained from B by substituting the propositional variables $a_1, a_2, \ldots, a_n$ in all possible ways.

DEFINITION 1. Let L be an intermediate logic $LJ + A_1 + A_2 + \ldots + A_m$. If there holds the relation that, for any formula A which is consisted of the propositional variables $a_1, a_2, \ldots, a_m$,

$A \in L$ if and only if $\Pi_n(A_1), \Pi_n(A_2), \ldots, \Pi_n(A_m) \to A \in LJ$,

then we say that the set of axioms $\{A_1, A_2, \ldots, A_m\}$ for L has SSP.

DEFINITION 2. For a logic L, if there is a set of axioms $\{A_1, A_2, \ldots, A_m\}$ with SSP such that $L = LJ + A_1 + A_2 + \ldots + A_m$, then we say that L has SSP.

We remark that the classical logic, the logic LQ $(LJ + \neg A \vee \neg\neg A)$, the n valued logics S_n defined in Gödel [1], Gödel infinitely many valued logic S_w, and LJ have SSP. (Cf. [4], [6], [7])

In this paper, we prove that every finite logic defined by a finite Pseudo-Boolean algebra has SSP.

2 Axioms of finite logics

By $L(M)$, we mean the logic defined by a finite Pseudo-Boolean algebra M.

In [2] has been given an example of axioms to be added to LJ in order to obtain $L(M)$ in the axiomatic way. Here we cite the method to construct the additional axioms for $L(M)$. The new axioms to be added are Z_M and X_n as defined below, where n is a number of truth values of M.

We suppose that $V = \{v_1, v_2, \ldots, v_n\}$ is the set of truth values of M, and v_1 is the designated value.

$P_k = \{a_1, a_2, \ldots, a_k\}$ $(k = 1, 2, \ldots)$.

$F = \{f \mid f : P_n \to V\}$.

For $f \in F, V(f) = \{f(x) | x \in P_n\}$.

$f^{-1}(v_i) = \{a_j | f(a_j) = v_i\}$.

$\phi_f(v_i)$ is the element with the least suffix of the set $f^{-1}(v_i)$.

For the logical connectives $\supset, \wedge, \vee$ and $\neg$, we call their truth tables as T_I, T_C, T_D, T_N, respectively.

Let W be a subset of V. We write the closure of W generated by the four logical connectives from the elements of W as W^c. Let be that $v_k = v_i \vee v_j$ and $v_k \notin W$, for example. Then we add $v_i \vee v_j$ to W instead of v_k to get W^c. In case that other combinations of W get the same value v_k, we add only one of them.

The part of T_I corresponding to $V(f)^c$ can be expressed by the following informal expression:

$$v_1 \supset v_1 = v_1 \text{ and } v_1 \supset v_2 \text{ and } \ldots \text{ and } v_i \supset v_j \text{ and } \ldots \text{ and } v_n \supset v_n = v_1.$$

Let $F_I(f)$ be the formula obtained from the above expression by substituting v_i by $\phi_f(v_i)$, $=$ by $\equiv$, "and" by $\wedge$. Similarly we define $F_C(f), F_D(f)$, and $F_N(f)$ corresponding to the logical connectives $\wedge, \vee$ and $\neg$.

Further, we define the following formulas.

$$E(f) = \bigwedge_{v_i \in V(f)} (\bigwedge_{a_j \in f^{-1}(v_i)} (\phi_f(v_i) \equiv a_j))$$

$$F(f) = F_C(f) \wedge F_D(f) \wedge F_N(f) \wedge F_I(f)$$

$$Z_M = \bigvee_{f \in F} (E(f) \wedge F(f))$$

$$X_n = \bigvee_{1 \leq i < j \leq n+1} (a_i \equiv a_j)$$

In [2], the logic $L(M)$ is axiomatized as $LJ + Z_M + X_n$, that is

$$L(M) = LJ + Z_M + X_n.$$

We call this axiomatic system as LM.

3 Finite logics and SSP

Now, the following theorem.

THEOREM 3. *$\{Z_M, X_n\}$ has SSP.*

PROOF. Let A be a formula and $a_1, a_2, \ldots, a_m$ are all the propositional variables contained in A. It is immediate that if $\Pi_m(Z_M), \Pi_m(X_n) \to A \in$

LJ, then $A \in LM$. Conversely, we prove along the proof of $LM \supseteq L(M)$ in [2]. We suppose $A \in L(M)$. And we divide two cases.

(1) The case $m \leq n$, we remark that $Z_M \to A \in LJ$ by the result in [2].

(2) The case $n < m$, suppose that G is the set of one-to-one mappings from P_{n+1} into P_m. It is obvious that G has only finite number of elements. By $X_n(g)$, where $g \in G$ we mean the formula obtained from X_n by substituting $a_1, a_2, \ldots, a_{n+1}$ by $g(a_1), g(a_2), \ldots, g(a_{n+1})$, respectively. In [2], $\bigwedge_{g \in G} X_n(g), Z_M \to A \in LJ$ is proved. Here we remark that $\bigwedge_{g \in G} X_n(g)$ and $\Pi_m(X_n)$ are equivalent in LJ.

COROLLARY 4. *Every finite logic has SSP.*

4 Other results

THEOREM 5. *If L and M are logics with SSP, then the logics $L \cap M$ and $L \cup M$ have SSP.*

PROOF. In Miura [5], the logic $(LJ + A) \cap (LJ + B)$ is axiomatized as $LJ + A \vee B$, where sets of propositional variables in axioms A and B are mutually disjoint. On the other hand, the logic $(LJ + C) \cup (LJ + D)$ is axiomatized as $LJ + C + D$. So by the definition of SSP, the Theorem is immediate.

As to the non-trivial infinite logics with SSP, only two logics have been known, that is, S_w and LQ. We remark that, by Theorem 5, the intersections of S_w (or LQ) and finite logics have SSP, which give other examples of infinite logics with SSP.

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