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A HYPOTHESIS ON THE FINITENESS OF GRAPHS FOR ŁUKASIEWICZ’S PRECOMPLETE LOGICS (GRAPHS FOR PRIME NUMBERS)

The article offers on the basis of functional characteristics of Łukasiewicz’s finite-valued logics a partition of a natural series of numbers into equivalence classes such that every class has one and only one prime number. This causes prime numbers to be represented as rooted trees. A hypothesis is suggested that such representation is finite for every prime number and necessary condition for it is specified.

1. Preliminaries

Let $M_{n+1}^L = \langle V, \sim, \rightarrow, \{1\} \rangle$ where $n \in N$ and $n \geq 2$, be Łukasiewicz’s $(n+1)$-valued matrix. That is, $V = \{0, 1/n, \ldots, n-1/n, 1\}$, $\sim x = 1-x$, $x \rightarrow y = \min(1, 1-x+y)$ and $\{1\}$ is the set of designated elements of $M_{n+1}^L$. The propositional Łukasiewicz’s logic $L_{n+1}$ is defined as a set of all tautologies of the matrix $M_{n+1}^L$ [4].

We denote the set of all matrix functions from $L_{n+1}$ by $L_{n+1}$. Let $P_{n+1}$ be the set of all $n+1$-valued functions defined on the set $V$. Then the set of functions $R$ is called functionally precomplete (in $P_{n+1}$) set if an addition to $R$ of a function $f \notin R$ forms the set $\{R, f\}$ functionally complete, i.e. if $\{R, f\} = P_{n+1}$. In [1] Bochvar and Finn have proved the set of functions $L_{n+1}$ is functionally precomplete in $P_{n+1}$ iff $n$ is a prime number. It is shown in [3] that there exists $n+1$-valued logic $K_{n+1}$ that has a non-empty set of tautologies iff $n$ is a prime number, in this case it is proved that $K_{n+1} = L_{n+1}$. (The proof of the theorem in [3, p. 67] contains a misprint: instead of $x \lor y = (x \rightarrow y) \lor (y \rightarrow x) = max(x, y)$ there should be $x \lor y = (x \lor y) \lor (y \lor x) = max(x, y)$.)
Thus in the first case an arbitrary prime number is defined by a precomplete set of functions of related logic and in the second case – by a non-empty set of tautologies of $K_{n+1}$-logic the functional properties of which agree with those of precomplete logic $L_{n+1}$. We offer here a representation of arbitrary precomplete logic $L_{n+1}$ by non-empty set of non-precomplete logics $L_{n+1}$ and this actually results in one more definition of a prime number, namely, in the form of rooted tree.

2. **Partition of a set of Łukasiewicz’s logics $L_{n+1}$ into equivalence classes**

It follows from [1] that if there are values $i/n$ from $V = \{0, 1/n, \ldots, n-1\}$ such that their numerator and denominator are not reciprocal prime numbers and $0 < i < n$ then these values are responsible for non-precompleteness of $L_{n+1}$. Therefore they must be remoted and the set $V$ must be reconstructed. This procedure must be repeated until $L_{p+1}$ is built where $p$ is a prime number. Hence it is necessary to find the number of $m$ in the row $1, 2, \ldots, n-1$ reciprocal with $n$ and to add 2 since $0 < i < n$. It follows from the definition of Euler’s function $\varphi(n)$ in [6] that $m$ is the value of $\varphi(n)$. A convenient formula for calculation of $\varphi(n)$ is contained in [2]:

$$\varphi(1) = 1, \varphi(pm) = p\varphi(m), \text{ if } p|m, \text{ and } \varphi(pm) = (p - 1)\varphi(m) \text{ if otherwise.}$$

The construction of precomplete logic $L_{p+1}$ from arbitrary logic $L_{n+1}$ is reduced then to reprocessing of an arbitrary number $n$ into $p$, the result of $\varphi(n)$ being added by 1 each time. Let $\varphi^*(n) = \varphi(n) + 1$. It should be noted that in virtue of properties of $\varphi(n)$, $\varphi^*(p) = p$, where $p$ is a prime number, that is $p$ is reprocessed into $p$. Thus we have an algorithm by which an arbitrary natural number $n$ is reprocessed into a prime number $p$ (consequently $L_{n+1}$ into $L_{p+1}$):

1. $n$
2. $\varphi^*_1(n) = n$, if $n = p$ or $\varphi^*_1(n) = m_1$ where $m_1 < n$.
3. $\varphi^*_2(m_1) = m_1$ if $m_1 = p_j$ or $\varphi^*_2(m_1) = m_2$, where $m_2 < m_1$.
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   .
4. $\varphi^*_{k-1}(m_{k-2}) = m_{k-2} = p_1$,

where $i > j > 1$. 

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Following to this algorithm, for example, Lukasiewicz’s logic $L_{139}$ is reconstructed into precomplete logic $L_{14}$, in this case $k = 4$. It follows from the above algorithm that $\phi^*_k(n)$ induces a partition of a set of logics $L_{n+1}$ into equivalence classes by the relation $\simeq$, where $L_{n_1+1} \simeq L_{n_2+1}$ iff $\phi^*_k(n_1) = \phi^*_k(n_2)$ where $1 \leq i, j \leq k$. Hence any equivalence class contains one and only one precomplete logic $L_{p+1}$. Let $X_{p+1}, \ldots, X_{p+1}, \ldots$ be equivalence classes where $p_\alpha$ is $\alpha$-th prime number. For example $X_{p_\alpha+1} = \{6, 9, 11, 13\}$. The question suggests itself: what is the power of every class $X_{p_\alpha+1}$?

3. Graphs for prime numbers

Since the partition of natural series of numbers forms the basis of partition of a set of logics $L_{n+1}$ into equivalence classes then there arises in connection with the question in view a problem of constructing class $X_p$ for an arbitrary prime number $p$. For this end we must define a function inverse to Euler’s function which is designated by $\phi^{-1}(m)$ and is defined by relation $\phi^{-1}(m) = \{n : \phi(n) = m\}$. For example, if $\phi(n) = 4$ then the above equation has only four solutions, that is $\phi^{-1}(4) = \{5, 8, 10, 12\}$. The properties of $\phi^{-1}(m)$ are investigated in [2] where the lower and the upper bounds for any non-empty set of values of $\phi^{-1}(m)$ are defined. It should be noted that value set of $\phi^{-1}(m)$ is empty for all odd values of $m > 1$ and also for many even values of $m$. What is essential is that in [2] the author offers an effective method of constructing the value set of $\phi^{-1}(m)$ using any $m$ which is the value of $\phi(n)$. In principle this makes it possible to construct an algorithm which builds according to any prime number its equivalence class $X_p$. The concept of algorithm consists in the following:

1. $p$ is a prime number.
2. $p - 1$
3. $\phi^{-1}(p - 1) = \{\nu_e\}_1 \cup \{\nu_0\}_1$, where $\{\nu_e\}_1$ is a set of even values and $\{\nu_0\}_1$ is a set of odd values. If $\{\nu_0\}_1$ is empty then the equivalence class is built as in the above example when $p = 5$. If otherwise then
4. Every $\nu_0$ is subtracted by 1, that is we have $\{\nu_0 - 1\}_1$.
5. $\phi^{-1}(\{\nu_0 - 1\}_1) = \{\nu_e\}_2 \cup \{\nu_0\}_2$.

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If elements of \( \{\nu_0 - 1\} \) are not the values of \( \varphi(n) \) then the corresponding equivalence class is built since then \( \varphi^{-1}(\{\nu_0 - 1\}) = \emptyset \). If the number of even numbers which are not the values of \( \varphi(n) \) were finite, then all classes \( X_p \) starting from some \( p \) would have infinite power. It follows from the result of [5] however that there exists an infinite set in even numbers which are not values of \( \varphi(n) \). Thus, the necessary condition for finiteness of every class \( X_p \) is found. The question on sufficiency is open to discussion.

The algorithm for the construction of equivalence classes \( X_p \) using arbitrary \( p \) gives us a way of representing prime numbers in the form of rooted tree which we designate by \( T_p \) where \( p \) is a root and the set of elements \( X_p \) is a set of tops. For example, let \( p = 13 \):

\[
\begin{array}{c}
69 \quad 92 \quad 138 \\
35 \quad 39 \quad 45 \quad 52 \quad 56 \\
70 \quad 72 \quad 78 \quad 84 \quad 90 \\
25 \quad 33 \quad 44 \quad 50 \quad 66 \\
21 \quad 26 \quad 28 \quad 36 \quad 42 \\
13
\end{array}
\]

Graphs for the first fifty prime numbers are finite. The hypothesis consists in that every rooted tree \( T_p \) is finite. Its proof would have a definite sense for number theory.

References


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