1. Introduction

This paper contains a logic enabling us to reason in the presence of vagueness phenomena. We consider an epistemological vagueness of concepts caused by the unavailability of total information about a continuous world which we describe in observational terms. Lack of information is manifested by the existence of borderline cases for concepts. Since we are unable to perceive concepts exactly, we cannot establish a sharp boundary between an extension of a concept and its complement.

Some results for reasoning about vague concepts are already known in the literature [1], [2], [3], [4], [10]. Usually, a semantics of vague concepts is developed within the theory of fuzzy sets or the theory of supertruth [9]. Following the idea that reasoning about vague concepts requires a special logic, we introduce a propositional language with sentential operators enabling us to define positive, negative, and borderline regions of any vague concept. A semantics of the language is refined within a framework of the theory of rough sets (Pawlak [8]). The admitted semantics provides means for defining extensions of concepts in a formal way. We show that these extensions reflect properly our intuition connected with vagueness. For example, the law of excluded middle is not valid on the level of extensions. Moreover, for any concept, the extensions of formulas representing its positive, negative, and borderline regions provide a partition of a universe of discourse.
2. Indiscernibility

Assume we are given a non-empty set which is treated as a universe of discourse. While observing $U$ we are usually unable to distinguish its individual elements that is we grasp not single objects but some classes of them. In each class there are elements which cannot be distinguished by means of obtainable tools of observation. In other words we distinguish objects up to a certain equivalence relation.

Any pair $S = (U, R)$ consisting of a universe $U$ and an equivalence relation $R$ on $U$ is said to be an approximation space (Pawlak [7]). Relation $R$ plays a role of indiscernibility relation for universe $U$.

Within the usual set-theoretic framework we identify extensions of concepts with subsets of a universe of discourse. Given a subset $K$ of universe $U$, we say that $K$ is definable in the approximation space $S = (U, R)$ if it can be represented as a union of some of the equivalence classes of relation $R$. Usually it is a case that not all the subsets of $U$ are definable in space $S$. Given a non-definable subset $K$ of $U$, our observation restricted by $R$ causes $K$ to be perceived as a fuzzy object. In fact we observe not the single set $K$ but a family of those sets which cannot be distinguished from $K$. It follows that there is no sharp boundary between set $K$ and its complement $U - K$ and hence $K$ is observed with some tolerance. The limits of this tolerance are determined by a pair of definable sets called upper and lower approximation of $K$, respectively, and defined as follows:

- an upper approximation $\overline{RK}$ of set $K$ in space $S$ is the least definable subset of $U$ containing $K$,
- a lower approximation $\underline{RK}$ of set $K$ in space $S$ is the greatest definable subset of $U$ contained in $K$.

It follows from the given definitions that the following equalities hold.

**Fact 2.1.**

(a) $\overline{RK} = \{x \in U: \text{there is an } y \in U \text{ such that } (x, y) \in R \text{ and } y \in K\}$,

(b) $\underline{RK} = \{x \in U: \text{for all } y \in U \text{ if } (x, y) \in R \text{ then } y \in K\}$.

This means that an element belongs to $\overline{RK}(\underline{RK})$ whenever it possibly (necessarily) belongs to $K$.

If a set $K$ corresponds to a certain concept then we can define positive, negative, and borderline instances of the concept as follows:
– an element \( u \in U \) is a positive instance of a concept represented by set \( K \) iff \( u \in R\overline{K} \),
– an element \( u \in U \) is a negative instance of a concept represented by set \( K \) iff \( u \in U - \overline{R}K \),
– an element \( u \in U \) is a borderline instance of a concept represented by set \( K \) iff \( u \in \overline{R}K - R\overline{K} \).

In the following we list some properties of approximations. Let \( X \) and \( Y \) be any subsets of universe \( U \). Then the following conditions are satisfied.

**Fact 2.2.**
(a) \( R(X \cup Y) = RX \cup RY \),
(b) \( X \subseteq R\overline{X} \),
(c) \( R\overline{RX} = \overline{RX} \),
(d) \( \overline{R\emptyset} = \emptyset \).

**Fact 2.3.**
(a) \( R(X \cap Y) = R\overline{RX} \cap R\overline{RY} \),
(b) \( R\overline{RX} \subseteq X \),
(c) \( R\overline{RX} = R\overline{RX} \),
(d) \( \overline{RU} = U \).

It follows that the algebra \( P(U) \) of all the subsets of \( U \) with the additional operation \( \overline{R} \) and \( R \) is a topological field of sets, where \( \overline{R} \) and \( R \) are closure and interior operation, respectively.

**Fact 2.4.**
(a) \( R\overline{RX} = \overline{R}(-X) \),
(b) \( R\overline{RX} = \overline{R}(-X) \),
(c) if \( X \subseteq Y \) then \( \overline{R}X \subseteq \overline{R}Y \) and \( R\overline{RX} \subseteq R\overline{RY} \).

Relation \( R \) induces equivalence relations \( eq_R, eq_{eq_R}, \) and \( eq_R \) (equal up to \( R \)) in the set \( P(U) \) defined as follows (Pawlak [8]):

\[ K_1 eq_R K_2 \text{ iff } \overline{R}K_1 = \overline{R}K_2, \]
\[ K_1 eq_{eq_R} K_2 \text{ iff } \overline{R}K_1 = \overline{R}K_2, \]
\[ K_1 eq_R K_2 \text{ iff } K_1 eq_{eq_R} K_2 \text{ and } K_1 eq_{eq_R} K_2. \]

These relations play a role of approximate equalities of sets. Relations \( eq_R \) can be treated as an indiscernibility relation in a family of universe \( U \). Given a non-definable subset \( K \) of \( U \), we are not able to “see” it exactly, our
observation restricted by relation $R$ causes $K$ to be perceived as a family of those sets which cannot be distinguished from $K$ by means of relation $eq_R$. Clearly, if a set $K$ is definable in space $(U, R)$ then $\overline{R}K = RK = K$.

**Fact 2.5.** The following conditions are equivalent:

(a) $K_1 eq_R K_2$,

(b) $K_1 \subseteq \overline{R}K_2$ and $R K_2 \subseteq K_1$.

3. Logic of vague concepts

In this section we define a formalized propositional language whose expressions are intended to represent concepts. Atomic concepts are represented by propositional variables (denoted by $p, q, \ldots$), and they are interpreted as subsets of a universe. Composed concepts are constructed from propositional variables by using the operations of negation ($\neg$), disjunction ($\lor$), conjunction ($\land$), implication ($\rightarrow$), and the special unary operations denoted by $pos$ (positive), $neg$ (negative), and $bor$ (borderline). A formula of the form $posA (negA, borA)$ is interpreted as a set of positive (negative, borderline) instances of the concept represented by $A$.

By a model we mean a pair $M = (S, m)$ consisting of an approximation space $S = (U, R)$ and a meaning function $m$ which assigns subsets of universe $U$ to propositional variables. Let $m(p)$ denote the set assigned to a variable $p$. We define inductively a satisfiability of formulas by non-empty subsets of set $U$. We say that set $K$ satisfies formula $A$ in model $M$ ($K sat_M A$) if the following conditions are satisfied:

- $K sat_M p$ iff $K \subseteq m(p)$,
- $K sat_M A \lor B$ iff $K sat_M A$ or $K sat_M B$,
- $K sat_M A \land B$ iff $K sat_M A$ and $K sat_M B$,
- $K sat_M A \rightarrow B$ iff non$K sat_M A$ or $K sat_M B$,
- $K sat_M \neg A$ iff non$K sat_M A$,
- $K sat_M posA$ iff for all non-empty $K' \subseteq U$ if $K eq_R K'$ then $K' sat_M A$,
- $K sat_M negA$ iff for all non-empty $K' \subseteq U$ if $K eq_R K'$ then for all non-empty $K'' \subseteq U$ if $K' \cap K'' \neq \emptyset$ then $K'' sat_M \neg A$,
- $K sat_M borA$ iff there is a non-empty $K' \subseteq U$ such that $K eq_R K'$ and $K' sat_M A$ and there is a non-empty $K'' \subseteq U$ such that $K eq_R K''$ and $K'' sat_M \neg A$. 

It follows from these definitions that a set $K$ satisfies a formula of the form $posA$ (negA, borA) whenever $K$ consists of positive (negative, borderline) instances of the concept represented by $A$. This logic was briefly described in Orłowska [6].

The admitted semantics enables us to define an extension of a concept represented by a formula $A$ in a model $M$ as follows:

$$ext_M A = \bigcup \{K : K \text{ sat } M A\}$$

Thus the extension of a formula in a model is a union of all the subsets of a universe of the model which satisfy the formula.

In the following we list some facts which follow from the given definition.

**Fact 3.1.**
(a) $ext_M p = m(p)$ for any propositional variable $p$,
(b) $ext_M \neg A \subseteq -ext_M A$,
(c) $ext_M (A \lor B) = ext_M A \cup ext_M B$,
(d) $ext_M (A \land B) \subseteq ext_M A \cap ext_M B$,
(e) $ext_M (A \rightarrow B) = ext_M \neg A \cup ext_M B$.

Conditions (b) and (d) state that an extension of a negated formula and an extension of a conjunction of formulas behave non-classically. As a consequence the law of excluded middle does not hold on the level of extensions, namely we have the following theorem.

**Fact 3.2.**
(a) $ext_M A \cup ext_M \neg A \subseteq U$,
(b) $ext_M A \cap ext_M \neg A = \emptyset$.

Extensions of modal formulas satisfy the following conditions.

**Fact 3.3.**
(a) $ext_M posA = -ext_M borA$,
(b) $ext_M negA = ext_M A = -ext_M borA$,
(c) $ext_M borA = -ext_M posA \cap -ext_M negA = R_{ext_M} A - ext_M borA$,
(d) $ext_M (posA \lor borA) = R_{ext_M} A$.

Thus, the extension of a formula of the form $posA$ is the lower approximation of $ext_M A$ in the space $S$, that is $posA$ represents the set consisting
of those elements of the universe which definitely belong to $\text{ext}_M A$. The extension of a formula of the form $\neg A$ is the complement of the upper approximation of $\text{ext}_M A$, and hence $\neg A$ corresponds to the set of elements which definitely do not belong to $\text{ext}_M A$. Similarly, a formula of the form $\text{bor}A$ corresponds to the set of borderline instances of the concept represented by $A$.

**Fact 3.4.**

(a) $\text{ext}_M \text{pos}A \cup \text{ext}_M \text{bor}A \cup \text{ext}_M \neg A = U,$
(b) $\text{ext}_M \text{pos}A \cap \text{ext}_M \neg A = \emptyset,$
(c) $\text{ext}_M \text{pos}A \cap \text{ext}_M \text{bor}A = \emptyset,$
(d) $\text{ext}_M \neg A \cap \text{ext}_M \text{bor}A = \emptyset.$

It follows that sets $\text{ext}_M \text{pos}A, \text{ext}_M \neg A,$ and $\text{ext}_M \text{bor}A$ provide a partition of the universe $U$.

A formula $A$ is said to be true in a model $M = ((U, R), m)$ if there is a family $S$ of subsets of set $U$ such that $\bigcup S = U$ and for each $K \in S$ we have $K \text{sat}A$.

**Fact 3.5.** The following condition are equivalent:

(a) $A$ is true in model $M = ((U, R), m)$,
(b) $\text{ext}_M A = U$.

4. Extensions of the logic

The formalized language presented in the previous section can be extended in many ways. In this section we sketch some of its possible refinements.

Assume we are given a model $M = ((U, R), m)$. If we are interested in considering concepts up to the equivalence $\equiv_R$, then we may introduce the operations $\text{pos}$, $\text{neg}$, and $\text{bor}$ connected with the relation $\equiv_R$, namely:

$K \text{sat}_M \text{pos}A$ iff for all non-empty $K' \subseteq U$ if $K \equiv_R K'$ then $K' \text{sat}_M A$,
$K \text{sat}_M \neg A$ iff for all non-empty $K' \subseteq U$ if $K \equiv_R K'$ then for all non-empty $K'' \subseteq U$ if $K' \cap K'' \neq \emptyset$ then $K'' \text{sat}_M \neg A$,
$K \text{sat}_M \text{bor}A$ iff there is a non-empty $K' \subseteq U$ such that $K \equiv_R K'$ and $K' \text{sat}_M A$ and there is a non-empty $K'' \subseteq U$ such that $K \equiv_R K''$ and $K'' \text{sat}_M \neg A$. 
The similar operations might be defined for the equivalence $eq_R$. All these operations reflect an approximate view of concepts.

The second group of operations is connected with approximate inclusions and approximate intersections of sets. Given an approximation space $S = (U, R)$, we define the relations $\text{con}_R$ (top-contained), $\text{com}_R$ (bottom-contained) (Pawlak [8]), $\tau_R$ (top-connected), and $\xi_R$ (bottom-connected) in the family $P(U)$ as follows:

- $K_1 \text{con}_RK_2$ iff $RK_1 \subseteq RK_2$,
- $K_1 \text{con}_RK_2$ iff $RK_1 \subseteq RK_2$,
- $K_1 \tau_R K_2$ iff $RK_1 \cap RK_2 \neq \emptyset$,
- $K_1 \xi_R K_2$ iff $RK_1 \cap RK_2 \neq \emptyset$.

These relations enables us to add to our formalized language the modal operations of approximate necessity. Let $S$ be a family of subsets of universe $U$. Following the idea that in any approximation space we do not perceive single elements, we will regard elements of $S$ as possible worlds. Given a present world, past worlds are its approximate subsets and future worlds are its approximate supersets. We define three independent operations $\text{n}_f$ (necessary in the future), $\text{n}_p$ (necessary in the past), and $\text{n}$ (necessary) as follows:

- $Ksat_M \text{n}_f A$ iff for all $K' \in S$ if $K \text{con}_R K'$ then $K' sat_M A$,
- $Ksat_M \text{n}_p A$ iff for all $K' \in S$ if $K' \text{con}_R K$ then $K' sat_M A$,
- $Ksat_M \text{n} A$ iff for all $K' \in S$ if $K \tau_R K'$ then $K' sat_M A$.

These operations correspond to the approximate view of nondeterministic time scale, where the upper approximations of worlds are taken into account. The similar operations can be defined by using the bottom inclusion and the bottom connection of worlds. The respective logic can be treated as an extension of the tense logic defined in Orłowska [5].

If we are interested in handling a dynamics of perception or observation then we will consider a generalized approximation space $(U, \{R_i\}_{i \in I})$ consisting of a universe of discourse and a family of indiscernibility relations on set $U$, reflecting changes of abilities of observation. Then we introduce the families $\{pos_i\}_{i \in I}$, $\{neg_i\}_{i \in I}$ and $\{bor_i\}_{i \in I}$ of unary operations and define in a natural way a satisfiability of formulas in models of the form $M = (\langle U, \{R_i\}_{i \in I}, m \rangle$.
5. Summary

In this paper we introduced the logic for reasoning about vague concepts. The language was produced in which the distinctions between positive, negative, and borderline regions of any vague concept could be represented precisely. The definition and properties of extensions of vague concepts were developed. Our main idea was that reasoning about vague concepts should be classical but extensions of vague concepts should not obey the law of excluded middle. The presented logic can be a starting point for a further research on formalized theories of vague concepts.

References