INTERPOLATION PROPERTIES FOR A CLASS OF MANY-VALUED PROPOSITIONAL CALCULI

In the paper Weaver’s method (see [5]) is adapted to prove interpolation properties of many-valued propositional calculi standard in the sense of Rosser and Turquette. The case of $n$-valued Łukasiewicz calculi is discussed in connection with the results obtained.

1. Standard $n$-valued propositional calculi

Where $L = (L, F_1, \ldots, F_k)$ is an absolutely free algebra with the set $V(L)$ of free generators it will be referred to as a propositional language. Elements of $V(L)$ will be called propositional variables while elements of $L$ formulas. For any set $X \subseteq L$, the symbol $V(X)$ is used to denote the set of all propositional variables occurring in formulas of $X$.

Given a finite natural $n \geq 2$, let $U_n = (\{1, 2, \ldots, n\}, f_1, \ldots, f_k)$ be an algebra similar to $L$. Then, an $n$-valued matrix for $L$ is defined as the pair

$(1) \quad M_n = (U_n, I)$

with $I = \{1, 2, \ldots, s\}$ for some $1 \leq s < n$. We shall say that $M_n^I$ is standard, cf. [4], if in $U_n$ one can define two-argument operations $\Rightarrow, \lor, \land$ and one argument operations $\neg, J_1, J_2, \ldots, J_n$ such that

\begin{align*}
i \Rightarrow k & \notin I \quad \text{iff} \quad i \in I \quad \text{and} \quad k \notin I \\
i \lor k & \in I \quad \text{iff} \quad i \in I \quad \text{or} \quad k \in I \\
i \land k & \in I \quad \text{iff} \quad i \in I \quad \text{and} \quad k \in I \\
\neg i & \in I \quad \text{iff} \quad i \notin I \\
J_k(i) & \in I \quad \text{iff} \quad i = k
\end{align*}

for any $i, k \in \{1, 2, \ldots, n\}$. In the sequel the same symbols $\Rightarrow, \lor, \land, \neg, J_1, J_2, \ldots, J_n$ are also used to stand for propositional connectives corresponding to the above operations of standard matrices $M_n^I$.

Where $M_n^I$ is a standard $n$-valued matrix for $L$, we put $C_n^I$ to stand for the consequence operation determined by $M_n^I$ in $L$ in the usual way, i.e. for any $X \subseteq L, \alpha \in L$,

\[(2) \quad \alpha \in C_n^I(X) \text{ iff } \forall \text{ homomorphism (a valuation) } h : L \rightarrow U_n, h(\alpha) \in I \text{ whenever } h(\beta) \in I \forall \beta \in X,
\]

(2) Each pair of the form $S_n^I = (L, C_n^I)$ is what we call an $n$-valued standard propositional calculus.

Finally, let us recall that the calculi in question have a very important property expressed in the form of deduction theorem with respect to the (standard) implication connective, i.e. that

\[(\text{DED}) \quad \alpha \in C_n^I(X, \beta) \text{ iff } \beta \Rightarrow \alpha \in C_n^I(X),
\]

(DED) cf. [4].

2. Generalized Interpolation Lemma for standard propositional calculi

Throughout this section we shall deal with a given standard propositional calculus $S_n^I = (L, C_n^I)$, and consequently, $M_n^I$ and $C_n^I$ will also be treated as fixed. In turn for any finite set $X_f \subseteq L$, we put $\wedge X_f$ and $\vee X_f$ to stand for the standard conjunction and disjunction of all formulas of $X_f$ taken in arbitrary but fixed order.

Where $V$ is a finite set of propositional variables and $h$ a valuation of $L$ into $U_n$, let us put

\[(5) \quad R_n(V, h) = \wedge \{J_i(p) : p \in V \text{ and } h(p) = i \}
\]

and

\[K_n^I(\alpha, V) = \vee \{R_n(V, h) : h(\alpha) \in I \}
\]

LEMMA 1. For any finite set $V \subseteq V(L)$ and any $\alpha, \beta \in L$, if $C_n^I(\alpha) \neq L, \beta \in C_n^I(\alpha), V(\alpha) \cap V(\beta) \neq \emptyset$ and $V(\alpha) \cap V(\beta) \subseteq V$, then $\beta \in C_n^I(K_n^I(\alpha, V))$ and $K_n^I(\alpha, V) \in C_n^I(\alpha)$. 
PROOF. 1°. If for some \( h : \mathcal{L} \rightarrow \mathcal{U}_n \), \( h(\alpha) \in I \), then immediately from (4) we get \( h(K^I_n(\alpha, V)) \in I \) and therefore \( K^I_n(\alpha, V) \in C^I_n(\alpha) \).

2°. In turn, if for a valuation \( h : \mathcal{L} \rightarrow \mathcal{U}_n \), \( h(K^I_n(\alpha, V)) \in I \), then \( V(\alpha) \cap V(\beta) \subseteq V \) implies that \( h(K^I_n(\alpha, V(\alpha) \cap V(\beta))) \in I \). Consequently, due to (4) there is a valuation \( \overline{h} : \mathcal{L} \rightarrow \mathcal{U}_n \) with the following two properties:

i. \( \overline{h}(\alpha) \in I \)

ii. for every \( p \in V(\alpha) \cap V(\beta) \), \( \overline{h}(p) = h(p) \).

That allows to define a new valuation \( h^* \) as follows:

\[
h^*(p) = \begin{cases} \overline{h}(p) & \text{for } p \in V(\alpha) \\ h(p) & \text{for } p \in V(\beta). \end{cases}
\]

Notice that \( h^*(\alpha) \in I \). But, on the other hand, \( \beta \in C^I_n(\alpha) \). This implies that \( h^*(\beta) \in I \) from which we get \( h(\beta) \in I \), and therefore \( \beta \in C^I_n(K^I_n(\alpha, V)) \).

THEOREM 2 (Generalized Interpolation Lemma). For every finite \( V \subseteq V(\mathcal{L}) \) and for every \( X \subseteq \mathcal{L} \), \( \alpha \in \mathcal{L} \) such that \( V(X) \cap V(\alpha) \neq \emptyset \) and \( V(X) \cap V(\alpha) \subseteq V \)

\((\text{INT})\ \alpha \in C^I_n(X) \iff \text{there is a } \gamma \in \mathcal{L} \text{ with } V(\gamma) = V \text{ such that } \alpha \in C^I_n(\gamma) \text{ and } \gamma \in C^I_n(X)\).

PROOF: The implication from the right to left in (INT) is obtained immediately, cf.[2].

To prove the converse, assume that \( \alpha \in C^I_n(X) \). We discern between the following two cases: 1°. \( C^I_n(X) = L \), 2°. \( C^I_n(X) \neq L \).

1°. If \( C^I_n(X) = L \), then \( \gamma \) is defined as follows:

\[
\gamma = \neg J_1(p) \lor \ldots \lor J_n(p) : p \in V
\]

is a formula we are looking for - \( C^I_n(\gamma) = L \).

2°. Assume that \( C^I_n(X) \neq L \). \( C^I_n \) is determined by a finite matrix and therefore it is finite., cf.[2]. Thus the assumption that \( \alpha \in C^I_n \) implies that there is a finite set of formulas \( X_f \subseteq X \) for which \( \alpha \in C^I_n(X_f) \). In turn, both of the sets \( V(\alpha) \cap V(X) \), \( V(\alpha) \cap V(X_f) \) are finite, and moreover, the former is non-empty. Let us put
\[ V^* = (V(X) \cap V(\alpha)) - (V(X_f) \cap V(\alpha)) \]

and define \( X^*_f \) as follows:

\[
X^*_f = \begin{cases} 
X_f & \text{whenever } V^* = \emptyset \\
X_f \cup \{ p \Rightarrow p : p \in V^* \} & \text{otherwise.}
\end{cases}
\]

Clearly, \( C^I_n(X_f) = C^I_n(X^*_f) \) and since \( X^*_f \) is finite we can replace it by a single formula built up by the use of the conjunction connective - namely, by \( \beta^* = \bigwedge\{ \beta : \beta \in X^*_f \} \), for we have \( C^I_n(X^*_f) = C^I_n(\beta^*) \). Setting things right, we have \( \alpha \in C^I_n(\beta^*) \) and \( V(\beta^*) \cap V(\alpha) = V(X) \cap V(\alpha) \). Then to conclude the proof is suffices to make use os Lemma 1 and thus to put \( \gamma = K^I_n(\beta^*, V) \).

**Corollary 3.** If \( V(X) \cap V(\alpha) = \emptyset \) and \( \alpha \in C^I_n(X) \), then either \( C^I_n(X) = L \) or \( \alpha \in C^I_n(\emptyset) \).

**Proof:** \( S^I_n \) is uniform in the sense od [2] and therefore from [6] we get that for any \( X, Y \subseteq L \) and \( \alpha \in L \)

- If \( V(X) \cap V(\alpha) = V(Y) \cap V(\alpha) = \emptyset \), \( C^I_n(X) \neq L \) and \( \alpha \in C^I_n(X \cup Y) \), then \( \alpha \in C^I_n(Y) \).

Putting \( Y = \emptyset \) in (u) we obtain

- If \( V(X) \cap V(\alpha) = \emptyset \) and \( C^I_n(X) \neq L \) and \( \alpha \in C^I_n(X) \), then \( \alpha \in C^I_n(\emptyset) \)

From which our corollary follows easily.

Now, we are going to establish the \((\Rightarrow)-\)counterpart of Theorem 2.

**Theorem 2.** For every \( X \subseteq L \), \( \alpha, \beta \in L \) and for every finite subset \( V \subseteq V(L) \), if \( V(X \cup \{ \alpha \}) \cap V(\beta) \neq \emptyset \) and \( V(X \cup \{ \alpha \}) \cap V(\beta) \subseteq V \); then

\[
(\text{INT}_w) \alpha \Rightarrow \beta \in C^I_n(X) \iff \text{there is } \gamma \in L \text{ with } V(\gamma) = V \text{ such that } \alpha \Rightarrow \gamma \in C^I_n(X) \text{ and } \gamma \Rightarrow \beta \in C^I_n(\emptyset).
\]

**Proof:** Suppose that under the assumption of the theorem \( \alpha \Rightarrow \beta \in C^I_n(X) \). Then, using (DED) we get \( \beta \in C^I_n(X, \alpha) \) and therefore, by Theorem 2, there exists a formula \( \gamma \) such that \( V(\gamma) = V, \beta \in C^I_n(\gamma) \) and \( \gamma \in C^I_n(X, \alpha) \). Using again (DED) we obtain \( \alpha \Rightarrow \gamma \in C^I_n(X), \gamma \Rightarrow \beta \in C^I_n(\emptyset) \) which ends the proof of the "only if" part. An easy proof of the "if" part is omitted.
Corollary 5. If \( V(X \cup \{\alpha\}) \cap V(\beta) = \emptyset \) and \( \alpha \Rightarrow \beta \in C_n(X) \), then either \( C_n(X, \alpha) = L \) or \( \alpha \Rightarrow \beta \in C_n(\emptyset) \).

3. Interpolation within Łukasiewicz \( n \)-valued propositional calculi

P.S. Krzystek and S. Zachorowicz proved in [1] that \( n \)-valued Łukasiewicz calculi for \( n \geq 3 \) fail to have interpolation property with respect to the original implication connective \( \rightarrow \). On the other hand, each matrix \( M_n \) (including the case \( n = 2 \)) is standard, cf.[3], [4]. Consequently, all the results of the preceding section apply to the consequence operation \( C_n \) determined by \( M_n \)'s. In particular, \( C_n \) has the interpolation with respect to a propositional connective corresponding to the binary operation of \( M_n \) defined as follows:

\[
x \Rightarrow y = \begin{cases} 
1 & \text{whenever } x \neq 1 \\
y & \text{otherwise.}
\end{cases}
\]

cf.[4].

Observe also that \( \Rightarrow \) can be defined by the sole use of \( \rightarrow \):

\[
x \Rightarrow y = x \rightarrow_{n-1} y,
\]

\( (x \rightarrow_{n-1} y \) is a shorthand for \( x \rightarrow (x \rightarrow \ldots \rightarrow (x \rightarrow y) \ldots ) \) if \( n > 1 \) and \( y \) otherwise). Then the following assertion is an easy corollary to Theorem 4:

Assertion 6. For every finite \( V \subseteq V(L) \), every \( X \subseteq L \), \( \alpha, \beta \in L \), if \( V(X \cup \{\alpha\}) \cap V(\beta) \neq \emptyset \), \( V(X \cup \{\alpha\}) \cap V(\beta) \subseteq V \) and \( \alpha \rightarrow \beta \in C_n(X) \), then there is a \( \gamma \in L \) such that \( V(\gamma) = V \), \( \alpha \rightarrow_{n-1} \gamma \in C_n(X) \) and \( \gamma \rightarrow_{n-1} \beta \in C_n(\emptyset) \).

Clearly, we also get

Corollary 7. If \( V(X \cup \{\alpha\}) \cap V(\beta) = \emptyset \) and \( \alpha \in C_n(X) \), then either \( C_n(X, \alpha) = L \) or \( \beta \in C_n(\emptyset) \).
References


Institute of Philosophy
Lódz University

Institute of Philosophy and Sociology
Polish Academy of Science
Warszawa