WHAT IS THE EQUIVALENCE CONNECTIVE

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Dedicated to
Professor Stanislaw Surma
on his 50th Birthday

0.

An answer to the question posed in the title of this paper is to be understood as an attempt to put down basic conditions characterizing the equivalence connective.

1.

Let $F$ be the propositional language determined by an infinite set of variables and the usual connectives: $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence). The symbol $F^{\leftrightarrow}$ denotes the $\{\leftrightarrow\}$-reduct of the language $F$ i.e. the subset of $F$ containing all formulas with no connective but $\leftrightarrow$.

2.

An axiomatization of the $\{\leftrightarrow\}$-reduct of the classical consequence operation and the variety of algebra related in a natural way to the equivalential fragment of the classical propositional logic is know for quite a long time. These problems are also settled for intuitionistic propositional logic, some intermediate logic and the 3-valued logic of Łukasiewicz (see [4], [6], [3], [1]).

In this paper we shall define a certain consequence operation in $F$ characterizing the implication connective which we shall treat at the basic (comp. [2]). Next we strengthen our basic consequence by adding the obvious rules relating the equivalence and the implication. Finally we axiomatize the $\{\leftrightarrow\}$-reduct of the strengthened consequence and we present
a class of matrices being a natural semantic of the equivalential fragment of its logic.

3.

To define the basic consequence \( C \) in the language \( F \) we use the following inference rules:

(b1) \( \vdash \alpha \rightarrow \alpha \),
(b2) \( \alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash \alpha \rightarrow \gamma \),
(b3) \( \alpha \rightarrow \beta, \alpha \vdash \beta \),
(b4) \( \alpha \rightarrow \beta, \beta \rightarrow \alpha, \gamma \rightarrow \delta, \delta \rightarrow \gamma \vdash (\delta \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta) \).

The strengthened consequence \( C^* \) in the language \( F \) is achieved by adding to the rules (b1)-(b4) the following:

(b5) \( \vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta) \),
(b6) \( \vdash (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha) \),
(b7) \( \alpha \rightarrow (\beta \rightarrow \gamma), \alpha \rightarrow (\gamma \rightarrow \beta) \vdash \alpha \rightarrow (\beta \rightarrow \gamma) \).

4.

Let \( E^* \) be the consequence operation in the language \( F^{**} \) determined by the following rules of inference:

(e1) \( \vdash \alpha \rightarrow \alpha \),
(e2) \( \vdash (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha) \),
(e3) \( \alpha \rightarrow \beta, \alpha \vdash \beta \),
(e4) \( \alpha \rightarrow \beta, (\alpha \rightarrow \gamma) \vdash (\beta \rightarrow \gamma) \).

The fact below states that the consequence operation \( E^* \) reflects the sought after basic properties of the equivalence connective:

### Fact 1.

The consequence operation \( E^* \) is \( \{\leftrightarrow\}\)-reduct of \( C^* \), i.e. every \( X \subset F^{**}, E^*(X) = C^*(X) \cap F^{**} \).

The proof of this fact is based on the observation that the rules obtained from (b1),..., (b7) by replacing all occurrences of the implication by the equivalence can be deduced from the rules (e1),..., (e4).

5.
Professor R. Suszko in his paper [5] investigates a consequence operation $E$ in the language $F\leftrightarrow$ which characterizes his notion of the identity connective. It is significant that in order to obtain $E^*$ from $E$ one has only to replace the rule $\alpha \leftrightarrow \beta \vdash \beta \leftrightarrow \alpha$ with its stronger version $\vdash (\alpha \leftrightarrow \beta) \leftrightarrow (\beta \leftrightarrow \alpha)$. This elucidates the difference between the identity connective and the equivalence connective.

6.

By a weak equivalential algebra we mean an algebra $\mathfrak{A} = \langle A, \leftrightarrow \rangle$ of the type $\langle 2 \rangle$ satisfying the following conditions for every $a, b \in A$,

(a1) $a \leftrightarrow b = c \leftrightarrow c$ for some $c \in A$ iff $c = b$,
(a2) $a \leftrightarrow b = b \leftrightarrow a$.

The class of weak equivalential algebras will be denoted by $WE$.

Let $M$ be the class of all matrices $M = \langle \mathfrak{A}, A^0 \rangle$ where $\mathfrak{A} \in WE$ and $A^0 = \{ a \leftrightarrow a : a \in A \}$. Denoting by $E(M)$ the content of the class of matrices $M$ we obtain the following:

**Fact 2.** $E(M) = E^*(\emptyset)$.

7.

We say that two consequence operation in the same language are equivalentially undistinguishable if its $\{\leftrightarrow\}$-reducts are equal.

In [1] it is shown that the consequence operations of the 3-valued logic of Łukasiewicz and the 3-valued logic of Heyting are equivalentially undistinguishable.

Let $C^\Delta$ be the consequence operation in $F$ determined by the rules (b1), (b3), (b5), (b6), (b7) and the following:

$(\Delta^\Delta) \quad \alpha \rightarrow \beta, \gamma \rightarrow \delta \vdash (\delta \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)$.

Then we have:

**Fact 3.** $C^* < C^\Delta$ but they are equivalentially undistinguishable.

Let $C^\nabla$ be the consequence operation in $F$ determined by the following rules of inference:

$(\nabla 1) \vdash \alpha \rightarrow \alpha,$
$(\nabla 2) \vdash (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)),$
$(\nabla 3) \vdash (\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)),$
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(∇4) \( \alpha \rightarrow \beta, \alpha \vdash \beta \),

and (b5), (b6), (b7).

Conjecture. \( C^* \) and \( C^\triangledown \) are equivalentially undistinguishable.

References


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