ON FINITELY BASED CONSEQUENCE DETERMINED
BY A DISTRIBUTIVE LATTICE

Let \( F = \langle F, \land, \lor \rangle \) be a free algebra in the class of all algebras of type \( \langle 2, 2 \rangle \) freely generated by the set of variables \( V = \{ p_0, p_1, p_2, \ldots \} = \{ p_i : i \in \mathbb{N} \} \). The elements of the set \( F \) will be called formulas. We shall use the Latin lower case letters \( x, y, z \) for formulas and \( U, X, Y \) for sets of formulas. The symbol \( V(U) \) denotes the set of all variables occurring in formulas of \( U \). By \( R \) we denote the set of the following rules:

\[
\begin{align*}
R1. & \quad \frac{x \land y}{x}, & R7. & \quad \frac{x \lor (y \lor z)}{(x \lor y) \lor z} \\
R2. & \quad \frac{x \land y}{y \land x}, & R8. & \quad \frac{(x \lor y) \lor z}{x \lor (y \lor z)} \\
R3. & \quad \frac{x, y}{x \land y}, & R9. & \quad \frac{x \lor (y \land z)}{(x \lor y) \land (x \lor z)} \\
R4. & \quad \frac{x}{x \lor y}, & R10. & \quad \frac{(x \lor y) \land (x \lor z)}{x \lor (y \land z)} \\
R5. & \quad \frac{x \lor y}{y \lor x}, & R11. & \quad \frac{x \lor (y \lor z)}{(x \lor y) \lor (x \lor z)} \\
R6. & \quad \frac{x \lor (x \lor y)}{x \lor y}. & R12. & \quad \frac{x \lor x}{x}.
\end{align*}
\]

The set of rules \( R \) determines a consequence operation \( \text{Cn}_R : 2^F \to 2^F \) such that for every \( X \subseteq F \), \( \text{Cn}_R(X) \) is the smallest subset of \( F \) containing \( X \) and closed under the rules of \( R \).
Define the relation $\vdash_R \subseteq F \times F$ as follows:

$$x \vdash_R y \iff y \in Cn_R(\{x\})$$

for every $x, y \in F$.

Then we obtain the following

**Lemma 0.** For every $x, y, z \in F$:

(i) $(x \land y) \lor x \vdash_R x$,
(ii) $(x \land y) \lor z \vdash_R y \lor z$,
(iii) $(x \land y) \lor (x \land z) \vdash_R x \land (y \lor z)$.

**Proof.** (i)

(1) $(x \land y) \lor x \vdash_R x \lor (x \land y)$ \{R5\}
(2) $x \lor (x \land y) \vdash_R (x \lor x) \land (x \lor y)$ \{R9\}
(3) $(x \lor x) \land (x \lor y) \vdash_R x \lor x$ \{R1\}
(4) $x \lor x \vdash_R x$ \{R12\}.

As analogous to (i), the proofs of (ii) and (iii) are left out.

Let $K = \langle K, \cap, \cup \rangle$ be a distributive lattice with 0 and 1 such that $0 \neq 1$, and let $K = \langle K, \{1\} \rangle$. The class of all such matrices $K$ will be denoted by $V$. Every matrix $K \in V$ determines a consequence operation $C_K$ as follows:

$$x \in C_K(X) \iff \forall h \in Hom(F, K)[h(X) \subseteq \{1\} \Rightarrow h(x) = 1]$$

for every $x \in F$ and $X \subseteq F$, (cf. [1]).

In this paper we show that $Cn_R = C_K$, for every $K \in V$.

The following lemma holds.

**Lemma 1.** $Cn_R \leq C_K$, for every $K \in V$.

We omit an easy proof of this lemma.

Let $AE$ be a smallest set satisfying the following conditions:

a. $V \subseteq AE$
b. $x, y \in AE \Rightarrow x \lor y \in AE$.

Now, we define the function $d : F \to 2^F$ as follows:

a. $d(x) = \{x\}$, if $x \in V$,
b. $d(x \land y) = \{x \land y\}$,
c. $d(x \lor y) = d(x) \cup d(y)$,

for every $x, y \in F$. 
Then we obtain

**Lemma 2.** For any \( x \in F - AE \) there exist \( y, z \in F \) such that \( y \land z \in d(x) \).

The proof of this lemma is straightforward.

Now let \( s \) be the function \( s : F \to N \) defined as follows:

1. \( s(x) = 0 \), if \( x \in V \),
2. \( s(x \land y) = s(x \lor y) = s(x) + s(y) + 1 \),

for every \( x, y \in F \).

Then the following holds.

**Lemma 3.** If \( d(x) = \{ z_1, z_2, \ldots, z_n \} \) and \( n \geq 2 \), then for every permutation \( i_1, i_2, \ldots, i_n \) of the sequence \( 1, 2, \ldots, n \) we have:

1. \( C_{nR}(\{ x \}) = C_{nR}(\{ z_{i_1} \lor (z_{i_2} \lor \ldots \lor (z_{i_{n-1}} \lor z_{i_n}) \ldots) \}) \),
2. \( V(\{ x \}) = V(\{ z_{i_1} \lor (z_{i_2} \lor \ldots \lor (z_{i_{n-1}} \lor z_{i_n}) \ldots) \}) \),
3. \( s(z_{i_1} \lor (z_{i_2} \lor \ldots \lor (z_{i_{n-1}} \lor z_{i_n}) \ldots)) \leq s(x) \).

**Proof.** (i) The lemma 3(i) is obtained by applying the rules \( R_4, R_5, R_7, R_8 \) and \( R_5 - R_8 \).

Easy proofs of (ii) and (iii) can be omitted.

**Lemma 4.** For every \( x \in F \) there exists a finite set \( U \subseteq AE \) such that:

1. \( C_{nR}(\{ x \}) = C_{nR}(U) \),
2. \( V(\{ x \}) = V(U) \).

**Proof.** If \( x \in V \), then the lemma is obvious. Assume inductively that for every formula \( w \) such that \( s(w) < s(x) \) the considered lemma holds. Let \( x = y_1 \lor z_1 \). If \( x \in AE \), then the lemma holds, for \( U = \{ x \} \). Let \( x \in F - AE \). Hence, by Lemma 2 and Lemma 3, for some \( y, z, x_1 \in F \) we have:

1. \( y \land z \in d(x) \),
2. \( C_{nR}(\{ x \}) = C_{nR}(\{ y \land z \} \lor x_1) \),
3. \( V(\{ x \}) = V(\{ y \land z \} \lor x_1) \),
4. \( s(y \land z) \lor x_1 \leq s(x) \).

From (2) and (4), applying the rules: \( R_9, R_{10}, R_{11}, R_1, R_2, R_3, R_5, \) and Lemma 0(iii), we obtain:
(5) \( Cn_R(\{x\}) = Cn_R(\{y \vee x_1, z \vee x_1\}) \),
(6) \( s(y \vee x_1) < s(x) \) and \( s(z \vee x_1) < s(x) \).

From (6) and from the inductive hypothesis, applying (5) we have:

(7) \( Cn_R(\{x\}) = Cn_R(U_1 \cup U_2) \),
(8) \( V(\{x\}) = V(U_1 \cup U_2) \),

for some finite subsets \( U_1, U_2 \) of \( AE \).

Thus the proof for \( x = y_1 \vee z_1 \) is completed. For \( x = y \land z \) it is quite simple.

**Lemma 5.** If \( x \in AE, U \subseteq AE \) and \( x \notin Cn_R(U) \), then there exists a set \( Y \subseteq V \) such that:

(i) \( x \notin Cn_R(Y) \),
(ii) \( U \subseteq Cn_R(Y) \).

**Proof.** Let \( x \in AE, U \subseteq AE \) and

(1) \( x \notin Cn_R(U) \).

Put

(2) \( V_z = V(\{z\}) - V(\{x\}) \), for every \( z \in U \).

Then, by (1), we have

(3) \( V_z \neq \emptyset \), for every \( z \in U \).

Hence, for \( Y = \{p_z : z \in U\} \), where \( p_z \) is some natural number such that \( p_z \in V_z \), we obtain:

(4) \( x \notin Cn_R(Y) \) and \( U \subseteq Cn_R(Y) \).

This completes the proof of Lemma 5.

Now, let \( K_2 = \langle \{0, 1\}, \cap, \cup \rangle \) and \( K_2 = \langle K_2, \{1\} \rangle, 0 \neq 1 \). We shall prove the following

**Theorem.** \( Cn_{K_2} \leq Cn_R \).

**Proof.** Let \( X \subseteq F \) and

(1) \( z \notin Cn_R(X) \).

According to Lemma 4, for some finite set \( U \subseteq AE \) we have:

(2) \( Cn_R(\{z\}) = Cn_R(U) \).
(3) $V(\{z\}) = V(U)$.

Hence for some $x_1 \in U$

(4) $x_1 \notin Cn_R(X)$.

By Lemma 4 we also have

(5) $Cn_R(X) = Cn_R(U_1)$, for some $U_1 \subseteq AE$.

From (4), (5) and Lemma 5

(6) $x_1 \notin Cn_R(Y_1)$ and
(7) $X \subseteq Cn_R(Y_1)$,

for some $Y_1 \subseteq V$.

Now, we define the function $v : V \to \{0, 1\}$ as follows:

(8) $v(p_i) = \begin{cases} 1, & \text{if } p_i \in Y_1 \\ 0, & \text{if } p_i \in V - Y_1. \end{cases}$

By Lemma 1 and by the fact that $R3, R4, R5 \in R$ we have

(9) $y \in Cn_R(Y_1) \iff h^v(y) = 1$, for every $y \in F$,

where $h^v$ is the extension of the function $v$ to the homomorphism of the algebra $E$ to $K_2$.

According to (2), (4) and (6) we obtain

(10) $z \notin Cn_R(Y_1)$.

Hence, by (9) and (7): $h^v(X) \subseteq \{1\}$ and $h^v(z) = 0$, which completes the proof of the theorem.

¿From Lemma 1 and from Theorem we have

COROLLARY. $Cn_R = C_\mathcal{K}$, for every $\mathcal{K} \in \mathcal{V}$.

References


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