M. W. Bunder

PARACONSISTENT COMBINATORY LOGIC

A logic \( L \), defined over a set of formulas \( F \), is said to be \textit{paraconsistent}\(^1\) if its set of theorems \( S \) contains \( X \) and \( \neg X \) for some \( X \) but \( F \neq S \).

The best known systems of this type are those of da Costa [5], which are based on the following principles:

1. \( \neg (A \& \neg A) \) should not in general be valid.
2. From \( A \) and \( \neg A \), it should not be possible, in general, to deduce an arbitrary formula.
3. Each calculus must contain the schemata and deduction rules of the classical propositional calculus \( C_0 \), that do not interfere with (1) and (2).

The axioms and deduction rules for the simplest of da Costa’s systems are those for the intuitionistic positive propositional calculus together with:

\[ \vdash \neg \neg A \supset A, \]
\[ \vdash A \lor \neg A, \]
\[ \vdash B^{(l)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)), \]
and
\[ \vdash A^{(l)} \& B^{(l)} \supset (A \supset B)^{(l)} \& (A \& B)^{(l)} \& (A \lor B)^{(l)} \]

where \( A^{(l)} \) is \( \neg (A \& \neg A) \).

Illative combinatory logic provides us with a number of paraconsistent logics that do not follow this pattern and which only satisfy (some of) the 3 conditions in a certain sense. Within it (see for example [2]) negation usually defined by:

\[ \neg X = X \supset \Xi HI \]

where \( \Xi HI \) has the property:

\(^1\)For more details on paraconsistent systems and for an extensive list of references, see [1].
\[ \Xi HI, \, HX \vdash X, \]

and "HY" stands for "Y is a proposition". (Note that H will be interpreted both as the predicate "is a proposition" and also as the class of all propositions.)

In [2] all the axioms of intuitionistic propositional calculus are derived, with the restriction that all the propositional variables are elements of H. This is done using axioms involving mainly \( \Xi \) and H.

Thus for example we have

\[ HX \vdash \neg(X \& \neg X), \]

and as certain terms cannot be shown to be in H, \( \neg(X \& \neg X) \) is not generally valid.

From X and \( \neg X \) it is possible to deduce an arbitrary proposition, but not an arbitrary formula, unless certain extra axioms are introduced\(^2\). If instead axioms allowing the proof only of \( \vdash \Xi HI \) were added to those of [2], the system would become paraconsistent as for every proposition X both \( \vdash X \) and \( \vdash X \supset \Xi HI \) would become provable.

Requirement (3) in contrast to (1) and (2) is not satisfied in the logic of [2]. We do have all theorems of the intuitionistic propositional calculus that involve variables that range over H.

A different system of combinatory logic that is in a sense paraconsistent is the system \( QD \) of Fitch (see [6]). [4] shows that within it we can prove for an arbitrary \( \alpha \) and a certain X that:

\[ \vdash X \quad \text{and} \quad DX \vdash X \supset \alpha, \]

where in Fitch’s system “DX” stands either for “X is a proposition” or “X satisfies strong excluded middle”.

Choosing \( \alpha \) to be \( \sim T \), where T is any theorem, we can easily derive (as for all Y, \( \vdash D(DY) \)): \( \vdash \sim DX \).

Now we may be really interested only in the propositions Y such that \( \vdash Y \). For any Y not in that category (since \( \vdash D(DY) \)) we have \( \vdash \sim DY \lor \sim Y \), which we might like to write as \( \vdash \sim Y \).

\(^2\)If the system has a term Q for equality with the properties:

\[ \vdash H(QXY) \quad \text{and} \quad QXY, X \vdash Y, \text{ for all } X \text{ and } Y. \]

every formula is derivable from X and \( \neg X \).

Alternatively, if \( \vdash H^{\Xi}X \) for all X, holds for some n, every formula is derivable (see [3]).
Thus we have $\vdash X$ and $\vdash \neg X$ from which no conclusions (other than ones such as $(X \& \neg X)$) can be drawn because we do not have $\vdash DX$. As in the system of [2] most propositional calculus results have some or all of their variables restricted to being in $D$.

Systems which are paraconsistent because all the propositions but not all the formulas (or terms) are of course not very interesting.

We could consider a class $J$ such that

$$\vdash \Xi IJ \quad \text{and} \quad \vdash \Xi JH$$

i.e., a class containing all of the classical (or intuitionistic) theorems but not all propositions.

We can then define $\neg A = (A \supset \Xi JJ)$ where $\vdash H(\Xi JJ)$, and have $\neg A, A, JX \vdash X$ so that only a fixed set of propositions (some not theorems of intuitionistic logic\(^3\)) can be derived from a contradiction\(^4\).

We can, using the deduction theorem of [2] prove $HA \vdash \neg(A \& \neg A)$, so again $\neg(A \& \neg A)$ is not always valid, but only restricted versions of the usual propositional calculus results hold.

It is also possible to have a different class (possibly empty) of propositions that are non theorems of intuitionistic (or classical) logic resulting from each contradiction.

If we define

$$\neg A = A \supset \Xi JA I$$

where $\vdash H(\Xi JA I)$, then the elements of $JA$ are the ones that result from $\vdash A$ and $\vdash \neg A$.

Logics with negations such as the ones immediately preceding are almost certainly not dependent on combinatory logic, although they can be represented very conveniently within it. We could conceivably have logics, without the wide range of potential (well formed) formulas given by combinatory logic, which satisfy (1) and (3) in full and (2) in that only a certain proper subset of the set of all non valid propositions is provable from theorems of the type $\vdash A \& \neg A$.

\(^3\)If the theory were classical, these would be tautologies.

\(^4\)J could be $K(T_1 \& T_2 \& \ldots \& T_n)$ for propositions $T_1, T_2, \ldots, T_n$ or some infinite class.
References


The University of Wollongong
Department of Mathematics
Wollongong, N.S.W.
Australia