ON UNIVERSAL ALGEBRAIC LOGIC AND CYLINDRIC ALGEBRAS

This is an abstract of the dissertation [1] which solved some problems raised in [2]. The subject is General Algebraic Logic in the sense of Rasiowa [5], but now for first order logics. Here we discuss the algebraic problems; their connections with (nonclassical and classical) logics were explained in [2]. The variety of cylindric algebras [4] was introduced for the classical first order logic; the present general algebraic (universal algebraic) approach is a generalization of the theory of that variety [4] to make it applicable to other first order logics as well (cf. Freeman [6]).

Throughout, $\alpha, \beta, \gamma$ denote infinite ordinals, $\omega$ is the set of natural numbers, and Ord is the class of finite ordinals.

**Definition 1.1:**

1. By a *type-scheme* we understand a quadruple $t = (T, \delta, \tau, c)$ where $T$ is a set, $\delta : T \to \omega$, $\tau : T \to \omega$, $c \in T$ and $\delta(c) = \tau(c) = 1$.

2. A type-scheme $t$ defines a *similarity type* $t_\alpha$ for each infinite ordinal $\alpha$ as follows:

   $t_\alpha : \Omega_\alpha \to \omega$, where the set $\Omega_\alpha$ of operation symbols is:

   $\Omega_\alpha = d \{ f_{i_1, \ldots, i_n} : f \in T, i_1, \ldots, i_n \in \alpha, n = \delta(f) \}$ with arities:

   $t_\alpha(f_{i_1, \ldots, i_n}) = d \tau(f)$.

   (Here $f_{i_1, \ldots, i_n}$ stands for the $n + 1$-tuple $(f, i_1, \ldots, i_n)$.)

**Example ([4]):** The similarity type of $\alpha$-dimensional cylindric algebras is:

$t_\alpha = \{ (\cdot, 2), (-, 1), (c_i, 1), (d_{ij}, 0) : i, j \in \alpha \}$.

The “cylindric type-scheme” consists of $T = \{ -, c, d \}$ and $\delta(\cdot) = \delta(\cdot) = 0$, $\delta(c) = 1$, $\delta(d) = 2$; $\tau(c) = 2$ etc.

The universe of an algebra $A$ is denoted by $A$. 
Definition 1.3. ([4], D. 2.6.1): Let $t$ be a type-scheme, $A$ an algebra of type $t_\alpha$, and let $\xi : \beta \rightarrow \alpha$ be arbitrary. Now, $Rd^\xi A$ is a new algebra of type $t_\beta$ obtained from $A$ as follows:

The universe of $Rd^\xi A$ is $A$.

The interpretation of the operation symbol $f_{i_1\ldots i_n} \in \Omega_\alpha$ in the new algebra coincides with the interpretation of $f_{\xi(i_1)\ldots \xi(i_n)}$ in the old one, i.e. $f_{\xi(i_1)\ldots \xi(i_n)}^{Rd^\xi A} =^d f_{i_1\ldots i_n}^A$.

$Rd^\xi A$ is called a generalized reduct of $A$ along $\xi$.

If $K$ is a class of algebras of similarity type $t_\alpha$, then $Rd^\xi K = \{Rd^\xi A : A \in K\}$.

An element $b \in A$ of an algebra $A$ of type $t_\alpha$ is said to be sensitive to the index $i \in \alpha$ if $b$ is not a fixed point of the operation $c_A^i$ (i.e. $c_A^i(b) \neq b$).

Definition 1.5 ([4], D. 1.11.1.): An algebra is locally finite dimensional if each of its elements is sensitive to finitely many indices only, i.e. if $(\forall a \in A)((\{i \in \alpha : c_A^i(a) \neq a\} \text{ is finite})$. An algebra is dimension complemented if to any finite subset $B$ of its universe there are infinitely many indices to which no element of $B$ is sensitive.

Definition 1.6 ([4], D. 2.6.28): Let $\alpha \leq \beta$ (i.e. $t_\alpha \subseteq t_\beta$). Let $B$ be an algebra of type $t_\beta$, and let $B'$ be its reduct of type $t_\alpha$ (i.e. we omit the operations which have indices greater than $\alpha$). An algebra $A \subseteq B'$ is said to be a neat subreduct of $B$ if the elements of $A$ are not sensitive in $B$ to the indices greater than $\alpha$, i.e. if $(\forall a \in A)(\forall i \geq \alpha)c_B^i(a) = a$.

If $K$ is a class of algebras of similarity type $t_\beta$, then $SNr_{t_\alpha}K$ denotes the class of those neat subreducts of elements of $K$, which are of type $t_\alpha$.

Definition 3.2: By a system of varieties of type-scheme $t$ we mean a sequence $\langle V_\alpha \rangle_{\alpha \in \text{Ord}}$, for which the following 1.-3. hold:

1. $V_\alpha$ is a variety of type $t_\alpha$, for every $\alpha \in \text{Ord}$.
2. $Rd^\xi V_\alpha \subseteq V_\gamma$ for every inclusion $\xi : \gamma \rightarrow \alpha$.
3. For every pair of ordinals $\gamma \leq \alpha$ and algebra $A$ of type $t_\alpha$:
   If every generalized reduct of type $t_\gamma$ of $A$ is in $V_\gamma$, then the original algebra $A$ is in $V_\alpha$, too (i.e. $[(\forall \xi : \gamma \rightarrow \alpha)Rd^\xi A \in V_\gamma] \Rightarrow A \in V_\alpha$.
Notation: From now on $\langle V_\alpha \rangle_{\alpha \in \text{Ord}}$ stands for an arbitrary system of varieties belonging to some type-scheme $t$, and

$Vf_\alpha =_d \{ A \in V_\alpha : A$ is locally finite dimensional $\}$

$Vc_\alpha =_d \{ A \in V_\alpha : A$ is dimension complemented $\}$

$Vn_\gamma \alpha =_d SNr_\gamma V_\alpha$.

Theorem 3.7: $\omega$ is the least ordinal $\rho$ for which it is true that for every system of varieties and ordinal $\alpha$, the sequence $\langle Vn_\omega \alpha + \rho + \nu \rangle_{\nu \in \text{Ord} \cup \omega}$ is constant, i.e. $Vn_\omega \alpha + \rho = Vn_\omega \alpha + \rho + \nu$ for every ordinal $\nu$.

Notation: $Vn_\alpha =_d Vn_\omega \alpha$.

Notations: The letters $H, S, P, Pr, Up, Sd$ denote the operators of taking homomorphic images, subalgebras, direct product, reduced products, ultraproducts and sandwich-subalgebras (see [3]), respectively. That is, if $K$ is a class of algebras, then $HK$ denotes the class of all homomorphic images of elements of $K$, etc.

Remark: The operators $SdUp$, $SuP$, $Spr$, $HP$ are known to coincide with the formation of hulls axiomatizable by $\exists_2$-formulas ($\forall \exists$-formulas), by universal formulas, by universal Horn-formulas (quasi-identities), and by identities, respectively.

Theorem 3.14-3.18: (For any $\langle V_\beta \rangle_{\beta \in \text{Ord}}$ and any $\alpha$):

1. $Vf_\alpha \subseteq Vc_\alpha \subseteq Vn_\alpha = SP^r Vn_\alpha \subseteq V_\alpha$.

2. If $|\alpha| = \omega$, then
   $\text{SdUp} Vf_\alpha = \text{SdUp} Vc_\alpha$
   $SU PVf_\alpha = SU PVc_\alpha = SP^r Vf_\alpha = SP^r Vc_\alpha$
   $HSPVf_\alpha = HSPVc_\alpha$
   and only these equalities are valid, i.e. there is a system of varieties $\langle V_\beta \rangle_{\beta \in \text{Ord}}$ such that the classes $Vf_\alpha, Vc_\alpha, SPVf_\alpha, SPVc_\alpha, SdUpVf_\alpha, SU PVf_\alpha, HSPVf_\alpha, Vn_\alpha, HVn_\alpha, V_\alpha$, are all different from one another (for any countable $\alpha$).

3. If $\alpha \geq \omega^+$, then no equality is valid except
   $SU PVf_\alpha = SP^r Vf_\alpha$.
   There is a system of varieties $\langle V_\beta \rangle_{\beta \in \text{Ord}}$ for which the classes $Vf_\alpha, Vc_\alpha, SPVf_\alpha, SPVc_\alpha, SdUpVf_\alpha, SdUpVc_\alpha, SU PVf_\alpha, SU PVc_\alpha, HSPVf_\alpha, HSPVc_\alpha, Vn_\alpha, HVn_\alpha, V_\alpha$
   are all different from one another, i.e., for instance $HSPVf_\alpha \neq$
DEFINITION 4.1: A system of varieties \( \langle V_\alpha \rangle_{\alpha \in \text{Ord}} \) satisfies the “generating condition”, if in every algebra of \( V_\omega \), elements sensitive only to finitely many indices generate no element sensitive to all indices \( (i \in \omega) \). More precisely:

\[
(\forall A \in V_\omega)(\forall m \in \Omega_\omega) \left[ \text{if } a_1, \ldots, a_n \in A \text{ are sensitive only to finitely many indices, then } (\exists i \in \omega)c_i(m(a_1, \ldots, a_n)) = m(a_1, \ldots, a_n) \text{ in } A \right].
\]

THEOREM 4.5: Let the system of varieties \( \langle V_\alpha \rangle_{\alpha \in \text{Ord}} \) satisfy the generating condition. Now, for every \( \alpha \in \text{Ord} \):

\[
\begin{align*}
\text{SdUp}Vf_\alpha &= \text{SdUp}Vc_\alpha \\
\text{SUp}Vf_\alpha &= \text{SUp}Vc_\alpha = \text{Sp}^rVf_\alpha = \text{Sp}^rVc_\alpha = Vn_\alpha \\
HSPVf_\alpha &= HSPVc_\alpha = HVn_\alpha,
\end{align*}
\]

and only these equalities are valid, i.e., there is a system of varieties \( \langle V_\alpha \rangle_{\alpha \in \text{Ord}} \) satisfying the generating condition, such that the classes \( Vf_\alpha, Vc_\alpha, SPVf_\alpha, SPVc_\alpha, \text{SdUp}Vf_\alpha, \text{SUp}Vf_\alpha, HSPVf_\alpha, V_\alpha \) are all different.

REMARK: The cylindric algebras of [4] form a systems of varieties \( \langle CA_\alpha \rangle_{\alpha \in \text{Ord}} \) satisfying the generating condition; therefore Th. 4.5. applies. Surprisingly, the inequalities of Th. 4.5. also hold for them with the exception that \( HVn_\alpha = Vn_\alpha \) is true for cylindric algebras. The following problem is open also for cylindric algebras.

PROBLEM: 1. Find a system of varieties \( \langle V_\alpha \rangle_{\alpha \in \text{Ord}} \) and a \( \Sigma_2 \)-formula \( \varphi \) (i.e. \( \varphi \equiv \exists \forall \exists \eta(\exists \eta) \)) such that \( Vf_\alpha \models \varphi \) and \( Vc_\alpha \not\models \varphi \) for some countable \( \alpha \). (By Th. 3.14. \( Vf_\alpha \) and \( Vc_\alpha \) are equivalent w.r.t. \( \Pi_2 \)-formulas.)

2. Find \( \langle V_\alpha \rangle_{\alpha \in \text{Ord}} \) and a first order \( \varphi \) such that \( Vf_\alpha \models \varphi \) and \( Vc_\alpha \not\models \varphi \) for some countable \( \alpha \). What is the smallest prenex for \( \varphi \) (\( \Sigma_2, \Pi_3, \Sigma_3, \ldots \)).

3. Solve the above problems for varieties satisfying the generating condition (and for arbitrary \( \alpha \)).

References


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