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ACKERMANN, TAKEUTI, AND SCHNITT:
FOR HIGHER-ORDER RELEVANT LOGIC
(Abstract)

It is noted in [1] that there is a close relation between Gentzen-style
cut-rule and the admissibility of Ackermann’s rule $\gamma$ for relevant logics
and theories. Thus far, $\gamma$ has at most been proved, in [2], for first-order
relevant logics. (Related methods are applied, in [1], to yield a new proof
of elementary logic, the classical adaptation of the $\gamma$-techniques as refined
in [3] having been carried out by Dunn.)

It is time to move up; at the higher-order level, the classical admissi-
bility of Gentzen’s cut-rule is the basic conjecture of Takeuti, whose veri-
would be a proof of $\gamma$ for a suitable higher-order logic. Such logics are worth
development on their own; the relevant analysis of the proposition is the
keystone of the enterprise, as is clear already in [6] and will be clearer in [7];
the natural generalization is to the analysis of the propositional function.

However, it is by no means clear what the natural generalization of
relevant first-order logics is, on grounds examined in [3]. In particular, there
are difficulties over identity; without safeguards, at least on the Leibniz
definition of identity, one may prove even at the level $R2$ of second-order
relevant implication such apparent fallacies of relevance as $x = y \rightarrow z = z$.
(A natural proposal to cure these difficulties, which stems from a suggestion
by Urguhart, is examined in [3].)

Difficulties over identity are in fact difficulties over comprehension prin-
ciples. I.e., in formulating the system $R2$, we adopt all the analogues for
predicate quantifiers of the first-order principles of $RQ$ (including but re-
stricted to universal instantiation to predicate letters, though not to com-
 pound predicate expressions), together with some instances of the compre-
hension principle

\[ [C] \quad \exists F \forall x (Fx \leftrightarrow A) \text{, where } F \text{ is not free in } A. \]

(Analogous schemes are of course entertained for general \( n \)-ary \( F \), not excluding the case \( n = 0 \).)

Amazingly, however, the admissibility of \( \gamma \) in \( R2 \) is not affected, assuming the usual pure second-order vocabulary, by any reasonable choice of the instances of \([C]\) that are to be assumed as axioms. But, as we are already prepared to expect, the proof is considerably more complicated than in the first-order case. Essentially, the idea is as follows. As in [1], proof of \( \gamma \) reduces to a demonstration that, for every non-theorem \( A \) of \( R2 \), there is a normal \( R2 \)-theory that does not contain \( A \). Normality here is taken in quite a strong sense. A normal \( R2 \)-theory must contain all theorems of \( R2 \) (whatever choice we havemade among potential axioms \([C]\) and their \( n \)-ary analogues); moreover, it must respect all the connectives and quantifiers, being consistent and complete on negation, prime on disjunction, \( \omega \)-complete on universal quantifiers, and \( \exists \) prime on existential quantifiers.

In particular, this means that, if \( T \) is to be \( R2 \)-normal, it must contain, whenever it contains \( \exists FA \) for \( n \)-ary predicate letter, a theorem \( A(G) \), for and \( n \)-ary predicate parameter \( G \), with the dual condition on the universal predicate quantifier.

So suppose that \( A \) is a non-theorem of \( R2 \). By Henkin methods (to which we may apply a nice refinement set out by Belnap in [7], and independently by Gabbay), we may build a completely regular \( R2 \) theory \( T_{-A} \), which does not contain \( A \) and which satisfies all the conditions for normality except perhaps the requirement of negation-consistency. Next, we blow \( T_{-A} \) up into an equivalent theory \( T^*_{-A} \), by adding copies of the predicate parameters of \( T_{-A} \). How many copies we add of a given predicate parameter \( F \) (which, for simplicity, we take as monadic) is calculated as follows. Think of all the formulas \( Fa \), where \( a \) is an individual parameter. We may think of \( F \) itself as a certain function, determined by \( T_{-A} \), defined on all individual parameters and with values in the 3-valued truth-set \( \{t,n,f\} \).

Specifically, where \( Fa \) is in \( T_{-A} \) but its negation isn’t, we think of \( F \) as having the value \( t \) at \( a \); if both \( Fa \) and \( \neg Fa \) are in, we think of \( F \) as having the value \( n \) at \( a \); if \( \neg Fa \) is in but \( Fa \) isn’t in, we think of \( F \) as having the value \( f \) at \( a \). This exhausts every possibility, since \( T_{-A} \) is negation-complete. We want \( F \), of course, to be not three-valued but two-valued;
the value \( n \) arises only at points \( a \) at which \( T_A \) is (perhaps) inconsistent, which is what prevents \( T_A \) from having the normality we desire. So let us make a copy \( F_i \) of \( F \) for each function from the set of individual parameters into \( \{t,f\} \) which agrees with \( F \) wherever possible; i.e., where, considered functionally, \( Fa \neq n \). Clearly, this may involve making a lot of copies; e.g., if \( F \) takes \( n \) as value denumerably many times, we shall have to make continuum many copies \( F_i \). After doing all this copying, analogously for each predicate parameter, we now form \( T^*_A \) by temporarily undoing its effect; i.e., by adding as an extra axiom, for given \( F \) and each new \( F_i \), \( \forall x(Fx \leftrightarrow F_i x) \), and analogously for each \( n \)-ary \( F \).

Next, we use \( T^*_A \) to determine a metavaluation, in something like the sense of [6]. Specifically, our valuation rules, for the metavaluation \( v \) from all sentences of the language of \( T_A \) into \( \{t,f\} \), will be as follows. Again, in describing the metavaluation on anatomic formulas, we confine ourselves, for simplicity, to the case where we have an anatomic formula \( F_i a \). But \( F_i \), as we constructed it, has already been associated with a certain function from individual parameters into \( \{t,f\} \), which we may call \( f_i \). Then, simply, let \( F_i a \) be \( t \) on \( v \) just in case \( f_i(a) = t \), and otherwise let \( F_i a \) be \( f \) on \( v \). (Without loss of generality, we may assume that all predicate parameters are associated with such (in general, \( n \)-place) functions \( f_i \), completing the specification of \( v \) on atomic sentences.)

Now we define \( v \) on all formulas by the following recursive procedure. \( v(\neg A) = \neg v(A) \) and \( v(A \land B) = v(A) \land v(B) \) in the obvious truth-functional sense. Similarly, \( v(\forall x Ax) = t \iff v(Ap) = t \) for each individual parameter \( p \), and \( v(\forall F AF) = t \iff v(AP) = t \) for each predicate parameter \( P \). We may, of course, treat existential quantifiers and disjunction as defined. Finally, \( v(A \rightarrow B) = t \) just in case both \( A \rightarrow B \) is a theorem of \( T^*_A \) and either \( v(\neg A) = t \) or \( v(B) = t \). (This latter move, referring truth on \( v \) not merely to truth-values of parts but to reference also to membership in some background theory, is at the heart of the metavaluation technique as developed in [3].) By reasonably straightforward, though still somewhat tedious, inductive argument, we show that the set of truths on \( v \) is both a normal \( R2 \)-theory and a sub-theory of \( T^*_A \). The key point, as the reader may be amused to check, is that adding all those extra predicate parameters enables us to verify all instances of the comprehension scheme \([C]\) that we have selected as axioms of \( R2 \). Since, perhaps, we have not selected all such instances as axioms, he may also be amused to check how the non-axioms can perhaps turn out false on \( v \). At any rate, we have got ourselves...
a normal $R2$-theory without our arbitrary non-theorem $A$, after which $\gamma$
follows as an easy corollary. (Central, incidentally, to the reasoning above is
a form of the converse Lindenbaum lemma, for we have shown-dual to the
usual Lindenbaum lemma – that a certain complete though inconsistent
theory has a normal subtheory: a lemma which, carefully characterized,
may be shown to hold generally.)

We have taken $R2$ to be a second-order version of $RQ$. Similarly, we
may form a pure type theory $RT$ by adding $n$-ary predicate variables and
parameters at arbitrary types. Again, we have considerable freedom in
choosing comprehension axioms $[C]$, while still carrying out the argument
for the admissibility of $\gamma$ which was sketched above. (I think that the
argument still goes through when extensionality axioms are added also, as
I have informally convinced myself. But I have not carried it out, even
informally, for the case in which typed lambda-terms are present; i.e., where
the language is not merely categorial but lambda-categorial.)

A detailed version of the above considerations and arguments may be
found in [3]. Even more details will be presented in [7], or perhaps in a
possible third volume of that work.

References


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