THE NUMBER OF QUASIVARIETIES OF DISTRIBUTIVE LATTICES WITH PSEUDOCOMPLEMENTATION

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Following Grätzer [2] by a distributive lattice with pseudocomplementation we mean an algebra $A = \langle A, \land, \lor, \neg, 0_A, 1_A \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ such that $\langle A, \land, \lor, 0_A, 1_A \rangle$ is a bounded distributive lattice and for every $a \in A$, $\neg a$ is the pseudocomplement of $a$ (i.e. the greatest element of the set $\{ x : x \in A, x \land a = 0 \}$). Let $D$ be the class of distributive lattices with pseudocomplementation. It is known that $D$ is a variety (see [5]) and a nice characterization of the lattice of subvarieties of $D$ is to be found in [3] (it is a denumerable lattice dually isomorphic to the ordinal $\omega^+\cdot\omega$). In this paper we will prove that the family of all subsets of a denumerably infinite set ordered by the inclusion is isomorphic to a family of quasivarieties of distributive lattices with pseudocomplementation. This result is a solution of the problem 63 of Grätzer [2] because it yields that the number of quasivarieties of distributive lattices with pseudocomplementation is $2^{\aleph_0}$.

Let $T = \langle T, \land, \lor, \neg, 0, 1 \rangle$ be the free algebra of terms of type $\langle 2, 2, 1, 0, 0 \rangle$ free-generated by a denumerably infinite set of variables $V = \{ z_0, z_1, \ldots \}$. By an identity we mean an expression of the form $\alpha \equiv \beta$ where $\alpha$ and $\beta$ are terms of $T$. The symbol $\text{Id}$ denotes the set of all identities. By an implication we mean an expression of the form $X \rightarrow \alpha \equiv \beta$ where $\alpha \equiv \beta$ is an identity and $X$ is a finite (possibly empty) set of identities.

Given an algebra $A \in D$, a valuation in $A$ is an arbitrary homomorphism $v$ of the algebra $T$ into $A$. An identity $\alpha \equiv \beta$ is satisfied by the valuation $v$ iff $v(\alpha) = v(\beta)$. The symbol $\text{Id}(v)$ denotes the set of all identities that are satisfied by $v$ and $\text{Id}(A) = \bigcap \{ \text{Id}(v) : v \text{ is a valuation in } A \}$. An implication $X \rightarrow \alpha \equiv \beta$ is satisfied by the valuation $v$ iff $X \subseteq \text{Id}(v)$.
implies that $\alpha \equiv \beta \in Id(v)$. The symbol $Im(v)$ denotes the set of all implications that are satisfied by $v$ and $Im(A) = \bigcap(Im(v) : v$ is a valuation in $A$).

A class $K$ of algebras of the same type is a variety iff for some set of identities $X$, $K = \{A : X \subseteq Id(A)\}$. The class $K$ is a quasivariety iff for some set of implications $Y$, $K = \{A : Y \subseteq Im(A)\}$. A characterization of quasivarieties of algebras was given by Malcev [4]. It should be noted that a quasivariety is closed under the formation of subalgebras and direct products (see [4]).

A convenient method of constructing distributive lattices with pseudo-complementation satisfying a prescribed set of implications can be obtained by transferring to lattice theory the following technique of forcing which is very familiar in logic.

Let $A = \langle A, \leq \rangle$ be a partially ordered set. A binary relation $\vdash \subseteq A \times T$ is called a forcing on $A$ iff for every $a, b \in A$, $\alpha, \beta \in T$ the following conditions hold (see [6]):

(i) For every $z \in V$, if $a \vdash z$ and $a \leq b$ then $b \vdash z$;
(ii) $a \vdash 1$;
(iii) $a \vdash 0$ ($\neg$ denotes the complement of $\vdash$);
(iv) $a \vdash \alpha \land \beta$ iff $a \vdash \alpha$ and $a \vdash \beta$;
(v) $a \vdash \alpha \lor \beta$ iff $a \vdash \alpha$ or $a \vdash \beta$;
(vi) $a \vdash \neg \beta$ iff for every $b \geq a$, $b \not\vdash \beta$.

**Lemma 1.** (see [6]).

(i) Every relation $\vdash \subseteq A \times V$ satisfying the condition (i) of the above definition can be extended (uniquely) to a forcing relation $\vdash$ on $A$.

(ii) For every forcing $\vdash$ on $A$, $a, b \in A$ and $\alpha, \beta \in T$ if $a \vdash \alpha$ and $a \leq b$ then $b \vdash \alpha$.

We say that an identity $\alpha \equiv \beta$ is satisfied by a forcing $\vdash$ on $A$ iff for every $a \in A$, $a \vdash \alpha$ iff $a \vdash \beta$. The symbol $Id(\vdash)$ denotes the set of all identities that are satisfied by $\vdash$ and $Id(A) = \bigcap(Id(\vdash) ; \vdash$ is a forcing on $A$). An implication $X \rightarrow \alpha \equiv \beta$ is satisfied by $\vdash$ iff $X \subseteq Id(\vdash)$ implies that $\alpha \equiv \beta \in Id(\vdash)$. The symbol $Im(\vdash)$ denotes the set of all implications that are satisfied by $\vdash$ and $Im(A) = \bigcap(Im(\vdash) ; \vdash$ is a forcing on $A$).

Following Birkhoff [1] we say that a partially ordered set is inductive iff every chain of its elements has an upper bound.
Lemma 2. If \( A = \langle A, \leq \rangle \) is inductive, \( a \in A \) and \( \alpha \in T \) then for every forcing relation \( \Vdash \) on \( A \) the following conditions are equivalent:

(i) \( a \Vdash \neg \alpha \),

(ii) for every maximal element \( b \in A \) such that \( a \leq b \), \( b \not\Vdash \alpha \).

For every partially ordered set \( A = \langle A, \leq \rangle \) let \( \Gamma(A) \) be the distributive lattice with pseudocomplementation of all hereditary subsets of \( A \) (see [2]). Recall that \( B \subseteq A \) is hereditary iff for every \( b \in B \), if \( a \in A \) and \( b \leq a \) then \( a \in B \). If \( H(A) \) is the family of all hereditary subsets of \( A \) then \( \Gamma(A) = \langle H(A), \cap, \cup, \neg, \emptyset, A \rangle \) where for every \( \Phi \in H(A) \), \( \neg \Phi = \bigcup(\Psi : \Psi \in H(A), \Psi \cap \Phi = \emptyset) \).

Lemma 3. \( \text{Im}(A) = \text{Im}(\Gamma(A)) \).

Let \( N \) be the set of all natural numbers. It will be convenient to identify a natural number \( n \) with the set of all natural numbers that are smaller than \( n \). For every \( n \subseteq N - \{0, 1\} \) we define the corresponding implication \( \Pi_n \) putting:

\[
\Pi_n = \neg
\bigvee (\neg z_i : i \in n) \equiv \bigvee (\neg z_i : i \in n) \Rightarrow 1 \equiv \bigvee (\neg (z_i \land \bigwedge (\neg z_j : j \in n - \{i\})) : i \in n).
\]

To explain what the implication \( \Pi_n \) says we need the following definitions. An element \( a \) of an algebra \( A \in \mathbb{D} \) is called skeletal (see [2]) iff for some \( b \in A \), \( a = \neg b \). A finite set \( B \) of elements of an algebra \( A \in \mathbb{D} \) is meet-independent iff for every \( C \subsetneq B \), \( \bigwedge C \neq \bigwedge B \). Now we can state the following:

Theorem 1. If \( A \in \mathbb{D} \) is such that the set of all non-unit skeletal elements of \( A \) can be extended to a proper ideal then the following conditions are equivalent:

(i) \( \Pi_n \in \text{Im}(A) \),

(ii) there is no meet-independent \( n \)-element set of skeletal elements of \( A \) whose join also is skeletal.

For every \( n \in N \) let \( P_n \) be the family of all \( n \)-element subsets of \( N \). For every \( I \subset N - \{0, 1\} \) let \( S_I = \bigcup(P_n : n \in I \cup \{1\}) \cup \{N\} \). Thus, for every \( I \subseteq N - \{0, 1\} \) we have the corresponding partially ordered set \( S_I = \langle S_I, \supseteq \rangle \).
It is obvious that all the maximal elements of $S_I$ are singletons from $P_1$, the smallest element of $S_I$ is $N$ and every ascending chain of elements of $S_I$ is finite which immediately yields that $S_I$ must be inductive.

**Lemma 4.** For every $I \subseteq N - \{0, 1\}$ and $n \in N - \{0, 1\}$ the following conditions are equivalent:

(i) $\Pi_n \in Im(S_I)$,

(ii) $n \in I$.

For every $I \subseteq N - \{0, 1\}$ we define a set of implications $\Pi(I) = \{\Pi_n : n \in I\}$ and the corresponding quasivariety $K(I) = \{A : A \in \mathcal{D}, \Pi(I) \subseteq Im(A)\}$. Applying Lemma 3 and Lemma 4 we get main result of this paper:

**Theorem 2.** For every $I, J \subseteq N - \{0, 1\}$, $K(I) \subseteq K(J)$ iff $I \supseteq J$.

**Proof.** Immediate, by Lemma 3 and Lemma 4. Q.E.D.

**Corollary.** There exist $2^{\aleph_0}$ of quasivarieties of distributive lattices with pseudocomplementation.

**References**


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