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ON THE EQUIVALENCE OF AJDUKIEWICZ-LAMBEK CALCULUS AND SIMPLE PHRASE STRUCTURE GRAMMARS

In [2], Bar-Hillel, Gaifman, and Shamir prove that the simple phrase structure grammars (SPG’s) defined by Chomsky are equivalent in a certain sense to Bar-Hillel’s bidirectional categorial grammars (BCG’s). On the other hand, Cohen [3] proves the equivalence of the latter ones to what he calls free categorial grammars (FCG’s). They are closely related to Lambek’s syntactic calculus which is, in turn, based on the idea due to Ajdukiewicz [1]. For some reasons, Cohen’s proof seems to be at least incomplete. Here, I give a short outline of the direct proof of the equivalence between SFG’s and FCG’s.

All necessary notions are defined in [2] and [3]. Small and capital letters denote categories and strings of categories, respectively. $X \rightarrow Y$ (resp. $X \rightarrow^* Y$) means “$X$ directly cancels to $Y$” (resp. “$X$ cancels to $Y$”). To the definition of direct cancellation in FCG’s correspond the following rules of cancellation:

1. $x/y^* y \rightarrow x$
2. $x/y^* y/z \rightarrow x/z$
3. $(x \cdot y)/z \rightarrow x \cdot (y/z)$
4. $x \rightarrow y/(x \cdot y)$
5. $y^* y \cdot x \rightarrow x$
6. $x/y^* y/z \rightarrow x/z$
7. $x/(y/z) \rightarrow (x/y)/z$
8. $x \rightarrow (y/x)\cdot y$

We are going to prove the two following facts:

**Theorem 1.** For any FCG $G$ there exists a SPG $H$ such that $L(G) = L(H)$.

**Theorem 2.** For any SPG $H$ there exists a FCG $G$ such that $L(G) = L(H)$.
The method of proof will follow that used in [2] for BCG’s instead of FCG’s. In comparison to [2], a difficulty arises here. Namely, in order for the vocabulary and the set of productions of the needed SPG to be finite, one must show that there are only finitely many categories occurring in all possible cancellations from strings of “vocabulary” categories to \( s \). This is obvious in case of BCG’s which admit only rules (1) and (1’) of cancellation. In our case, we deal with rules (4) and (4’) which increase the complexity of categories involved. Fortunately, it turns out that (4) and (4’) play a very restricted part in cancellations to primitive categories (in particular, to \( s \)). This will follow easily from Theorem 0.

For a given FCG, let \( U_0 \) be the set of all categories assigned to elements of the vocabulary. We define the sets \( U_1, U_2, U_3, U_4 \) as follows:

- \( U_1 \) is the set of constituents of \( U_0 \),
- \( U_2 \) results form \( U_1 \) by adding all categories of the form \( x/y \), \( x\backslash y \), \( (x\backslash y)/z \), \( x\backslash(y/z) \), \( x\backslash(x\backslash y) \), \( (y/x)\backslash y \) with \( x, y, z \) in \( U_1 \),
- \( U_3 \) results from \( U_2 \) by adding all categories of the form \( y/(x\backslash y) \), \( (y/x)\backslash y \) with \( x, y \) in \( U_1 \),
- \( U_4 \) results from \( U_3 \) in the same way that \( U_2 \) results from \( U_1 \).

It is easy to see that \( U_i \) are finite and closed with respect to constituents, and that \( U_{i-1} \subseteq U_i \) (\( i = 1, 2, 3, 4 \)). Moreover, the following facts hold:

**Lemma 1.** \( U_2 \) and \( U_4 \) are closed with respect to the rules (1) to (3’).

**Lemma 2.** If \( a_1 \ldots a_k \rightarrow^* b \), \( a_i \in U_0 \) (\( i = 1, \ldots, k \)), and whenever rule (4) or (4’) is applied, both \( x \) and \( y \) are in \( U_1 \), then \( b \in U_4 \).

We shall say that Operation 1L is applied to \( b \) to obtain \( c \) in a given direct cancellation if it has the form \( W^\bullet a^\bullet b^\bullet Z \rightarrow W^\bullet c^\bullet Z \) and \( a^\bullet b \rightarrow c \) is a substitution of the rule (1). If this is the case, we shall also say that operation 1R is applied to \( a \) to obtain \( c \). Similarly, we define operations 1’R, 1’R, 2R, 2’R, 2’L, 2’R, 3, 3’, 4, and 4’.

Operations 1’L, 2L, 2’L, 3, 3’ will be called Operations I. Operations 1L, 1R, 1’R, 2R, 2’R will be called Operations II.

For a given category \( x \), we shall denote by \( R(x) \) the set of all categories of the form \( o/x \) or \( o_1 \backslash (o_2/x) \).

**Lemma 3.** Operations I do not lead out of \( R(x) \).
**Lemma 4.** Let \( X \star a/c \rightarrow b/c \) and \( b/c \) results from \( a/c \) by Operations I only. Then \( X \star a \rightarrow b \) and \( X \star a/d \rightarrow b/d \). Moreover, the new cancellations contain the same number of applications of (4) and (4') as \( X \star a/c \rightarrow b/c \).

For a given category \( t \), we define the set \( PP(t) \) as the smallest set satisfying the following conditions:

(i) \( q/t \in PP(t) \) for any \( q \),
(ii) If \( x \in PP(t) \), then \( y \setminus x \in PP(t) \) and \( x/y \in PP(t) \).

Let \( X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_0 = r/t \). We define the sequence \( p_0, \ldots, p_n \) (\( m \leq n \)) of predecessors of \( r/t \) and the corresponding sequence \( P_0, \ldots, P_m \) of pairs of natural numbers by simultaneous induction in such a way that each \( p_i \) is a term of \( X_i \) and the following lemmas are true:

**Lemma 5.** Let \( p_i \) be a predecessor of \( r/t \). If \( P_i = (0,1) \), then \( p_i \) has the form \( q/t \). If \( P_i = (1,1) \), then \( p_i \) has the form \( c \setminus (q/t) \). In both cases, \( p_0 \) results from \( p_i \) by Operations I only.

**Lemma 6.** Each predecessor of \( r/t \) is in \( PP(t) \).

**Lemma 7.** If \( p_i \) is a predecessor of \( r/t \), \( P_i \neq (0,1) \), and \( p_i \) has the form \( x/y \), then \( x \) has a constituent of the form \( q/t \).

In order to be able to apply induction, we assign a natural number \( N(x) \) to each occurrence \( x \) of a category in a given cancellation in such a way that the following fact holds:

**Lemma 8.** If \( a_1 \ldots a_k \rightarrow^* W \star b \star c \star Z \rightarrow W \star d \star Z \), then \( N(b) < N(c) = N(d) \).

From “right” lemmas 3 to 8, their “left” variants, i.e., lemmas 3' to 8' are obtained, roughly speaking, by replacing everywhere \( X \star Y \) by \( Y \star X \) and \( x/y \) by \( y/x \). In that order, the notions of Operations I', Operations II', \( L(x) \), predecessors of \( t/r \), \( PP'(t) \), and \( N'(x) \) are defined analogously to those of Operations I, Operations II, \( R(x) \), predecessors of \( r/t \), \( PP(t) \), and \( N(x) \).

Let us consider all possible cancellations from a given \( X \) to a given \( Y \). Such a cancellation will be called **minimal** if it contains minimal number of applications of rules (4) and (4').
Theorem 0. If $X = a_1 \ldots a_k \rightarrow^* s$, $a_i \in U_0$ ($i = 1, \ldots, r$), then in any minimal cancellation from $X$ to $s$, whenever rule (4) or (4') is applied, both $x$ and $y$ are in $U_1$.

The proof of Theorem 0 is rather long and tedious; it involves all the lemmas (1) to (8'). Now, Theorem 1 may be proved in an analogous way to that used in [2], applying Theorem 0, Lemma 1, and Lemma 2. For Theorem 2, it was shown in [2] that for any SPG there exists an equivalent restricted categorial grammar. Consequently, it suffices to prove the following fact:

“For any FCG, if $U_0$ consists of categories of the form $x, x\backslash y, x\backslash(y\backslash z)$ with $x, y, z$ primitive, $X = a_1 \ldots a_k \rightarrow^* s$, $a_i \in U_0$ ($i = 1, \ldots, k$), then there exists a cancellation from $X$ to $s$ which involves only rule (1').”

This may be done by a slight modification of the proof of Theorem 0.

References


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