CHARACTERIZATION OF FINITELY AXIOMATIZABLE SETS ON THE BASIS OF A SYSTEM OF THE PROPOSITIONAL CALCULUS*

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The empty set and the set of all sentential variables are denoted by $\emptyset$ and $At$, respectively. The functions mapping the set $At$ into the set $\{0, 1\}$ are called valuations. We designate by $v|Z$ the function obtained from the valuation $v$ by restricting it to the set of variables $Z$. And analogously we designate by $K|Z$ the set of functions obtained from the valuations $v \in K$ by restricting them to the set $Z$.

Our considerations are devoted to the classical axiomatic system of the propositional calculus with the implication and negation connectives and the rule of detachment (Rasiowa [2]).

A set formulas $X$ is said to be finitely axiomatizable iff there exists a finite set of formulas $Y$ such that the set of all those formulas which can be inferred from axioms of the propositional calculus and from the set of formulas $Y$ by means of the detachment rule is equal to the set $X$.

Let $K$ be an arbitrary set of valuations. A set of formulas $X$ is called $K$-saturated (cf. Pogorzelski [1], the set $Sat_e$) iff $X$ is the set of all those formulas, which have the value 1 for all valuations belonging to the set $K$.

**Definition.** We say that a set of valuations $K$ satisfies the condition $W$ iff there exists sets of variables $Z_1, Z_2$ such that

$$Z_1 \cup Z_2 = At,$$

$$Z_1 \cap Z_2 = \emptyset,$$

*As abstract this article is not to be reviewed.*
the set $Z_1$ is finite, 
$K|Z_2 = \{0,1\}^{Z_2}$,
if $Z_1 \neq \emptyset$ then for every function $v \in K|Z_1$ and for every function $w \in K|Z_2$ there exists a valuation $u \in K$ such that
$$v = u|Z_1 \text{ and } w = u|Z_2.$$ 

**Theorem.** A consistent set of formulas $X$ is finitely axiomatizable iff there exists a set of valuations $K$ fulfilling the condition $W$ such that the set $X$ is $K$-saturated.

The proof of above Theorem yields a following simple procedure for finding axiom systems for some sets of formulas.

Let $X$ be a $K$-saturated set of formulas and let the set of valuations $K$ fulfills the condition $W$.

If $K = \{0,1\}^{A_1}$ then, of course, the empty set is the axiom system of the set $X$. Otherwise $Z_1 = \emptyset$. We denote then by $Z_3$ an arbitrary set of these variables $q \in Z_1$, which for arbitrary valuations $v, u \in K$ fulfill the condition $v(q) = u(q)$.

If the set $Z_1 - Z_3$ is non-empty, then we take the following notation:

$$Z_1 - Z_3 = \{p_1, p_2, \ldots, p_n\},$$

$$K(Z_1 - Z_3) = \{v_1, v_2, \ldots, v_s\},$$

$$p_1^i = p_i,$$

$$p_0^i = \sim p_i; \text{ for } i = 1,2,\ldots,n,$$

$$A_j = \begin{cases} 
    p_1^{v_j(p_i)} \Rightarrow (p_2^{v_j(p_2)} \Rightarrow \ldots \Rightarrow (p_n^{v_j(p_{n-1})} \Rightarrow \sim p_n^{v_j(p_n)})) \ldots \text{ if } n > 1 \\
    \sim p_1^{v_j(p_i)} \text{ if } n = 1; \text{ for } j = 1,2,\ldots,s,
\end{cases}$$

$$A = \begin{cases} 
    A_1 \Rightarrow (A_2 \Rightarrow \ldots \Rightarrow (A_{s-1} \Rightarrow \sim A_s) \ldots) \text{ if } s > 1, \\
    \sim A_1 \text{ if } s = 1.
\end{cases}$$

If the set $Z_3$ is non-empty, then we take the following notations:

$$Z_3 = \{q_1, q_2, \ldots, q_m\},$$

$$K|Z_3 = \{w\}.$$
A set $Y$ defined as follows:

\[
\{ A, q_1^{w(q_1)}, q_2^{w(q_2)}, \ldots, q_m^{w(q_m)} \} \text{ if } Z_1 - Z_3 \neq \emptyset \text{ and } Z_3 \neq \emptyset,
\]

\[
Y = \{ A \} \text{ if } Z_3 = \emptyset,
\]

\[
\{ q_1^{w(q_1)}, q_2^{w(q_2)}, \ldots, q_m^{w(q_m)} \} \text{ if } Z_1 - Z_3 = \emptyset,
\]

is the axiom system of the set $X$.

References
