Let $L = (L, \lor, \land, \sim)$ be a propositional language and $Cn$ a finitistic consequence operation which posses the following properties:

A1. $Cn(\{\alpha, \sim \alpha\}) = \mathcal{L}$,
A2. $Cn(\{\alpha\}) \cap Cn(\{\sim \alpha\}) = Cn(\emptyset)$,
A3. $Cn(\{\alpha \land \beta\}) = Cn(\{\alpha, \beta\})$,
A4. $Cn(X \cup \{\alpha \lor \beta\}) = Cn(X \cup \{\alpha\}) \cap Cn(X \cup \{\beta\})$.

One can define $\alpha \rightarrow \beta$ as abbreviation for $\sim \alpha \lor \beta$. It may be proved the

A5. $\alpha \rightarrow \beta \in Cn(X)$ iff $\beta \in Cn(X \cup \{\alpha\})$.

Thus, it is obvious that $Cn$ has all essential features of the consequence $Cn$ examined by Tarski in [4]. The aim of the paper is to show some interdependence and connections occurring between the consequence operation $Cn$ and its dual equivalent $dCn$ which was introduced by Wójcicki in [5].

Besides, we shall point to the relationship of $dCn$ to the consequence operation $Cn^{-1}$, defined by Slupecki in [2].

Let us recall that for every consequence $C$

\[ \alpha \in C^{-1}(X) \text{ iff there exists } \beta \in X \text{ such that } \beta \in C(\{\alpha\}) \]

Perhaps it is worthwhile to note that for every consequence $C$

\[ (1) \ \alpha \in C^{-1}(X) \text{ iff } \alpha \in dC(\{\beta\}), \text{ for some } \beta \in X \]

On the other hand it may be proved that

\[ (2) \ \alpha \in dCn(X) \text{ iff } \alpha \in Cn^{-1}(\{\beta_1 \lor \ldots \lor \beta_k\}) \text{ for some } \beta_1, \ldots, \beta_k \in X \]

Theorem 1. The following conditions are valid:

\[ dA0. \ dCn \text{ is a finitistic consequence operation}, \]
\[ dA1. \ dCn(\{\alpha, \sim \alpha\}) = \mathcal{L}, \]
\[ dA2. \ dCn(\{\alpha\}) \cap dCn(\{\sim \alpha\}) = dCn(\emptyset), \]
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\[ dA3. \ dCn(\{\alpha \lor \beta\}) = dCn(\{\alpha, \beta\}) \]
\[ dA4. \ dCn(\{\alpha \land \beta\}) = dCn(\{\alpha\}) \cap dCn(\{\beta\}). \]

Thus the system of concepts $(dCn, \land, \lor, \sim)$ is dual with respect to $(C_n, \lor, \land, \sim)$.

**Corollary 1.**

\[ dA5. \ \alpha \prec \beta \in dCn(X) \text{ iff } \beta \in dCn(X \cup \{\alpha\}), \text{ where } \alpha \prec \beta \text{ is the abbreviation for } \sim \alpha \land \beta. \]

The $C_n$–counterpart $A^{-1}0$ of $dA0$ is valid, i.e. $C_n^{-1}$ is a finitistic consequence operation. In general however $A^{-1}1, A^{-1}2, A^{-1}3, A^{-1}4, \text{ and } A^{-1}5$ cannot be proved. It follows from the fact that

**Assertion 1.** $C^{-1} \leq dC$, i.e. for every $X \subseteq L$, $C^{-1}(X) \subseteq dC(X)$.

The converse of Assertion 1 need not be true. Note yet that $C_n^{-1}(\{\alpha \lor \sim \alpha\}) = L$ and $C_n^{-1}(\{\alpha\}) \cap C_n^{-1}(\{\sim \alpha\}) = C_n^{-1}(\{\alpha \land \sim \alpha\})$ for every $X \subseteq L$.

**Theorem 2.** $ddCn(X) = Cn(X)$ for every $X \subseteq L$.

Once again this property is not possessed by $C_n^{-1}$. We have only $(C_n^{-1})^{-1} \leq C_n$.

Let $AX$, $KX$ and $NX$ be defined as [3]. The set $AX$ (the set $KX$) is the set of all disjunctions (conjunctions) formed by elements of $X$, $NX = \{\sim \alpha : \alpha \in X\} \cup \{\alpha : \sim \alpha \in X\}$. We additionally put $A\emptyset = NCn(\emptyset)$ and $K\emptyset = Cn(\emptyset)$.

**Theorem 3.** $C_n^{-1}(AX) = dCn(X)$ for every $X \subseteq L$.

**Theorem 4.**

(i) $dCn(ANX) = dCn(KNX)$,
(ii) $dCn(KNX) = dCn(ANX)$.

The counterparts of Theorem 3 for $C_n^{-1}$ are valid (cf. [3]).

**Theorem 5.**

(i) $Cn(X) = NdCn(NX)$,
(ii) $dCn(X) = NCn(NX)$. 
Proof. (i) Let $\alpha \in Cn(X)$. Hence exists a finite subset $X_f = \{\alpha_1, \ldots, \alpha_m\}$ of the set $X$ such that $\sim \alpha \in dCn(\{\sim \{\alpha_1 \wedge \ldots \wedge \alpha_m\}\})$. Thus $\sim \alpha \in dCn(\{\sim \alpha_1, \ldots, \sim \alpha_m\})$.

That implies $\alpha \in NdCn(NX)$. Now let $\alpha \in NdCn(NX)$. Since $NNdCn(Y) = dCn(Y)$, for every $Y \subseteq L$, then $\sim \alpha \in dCn(NX)$. So $\sim \alpha \in dCn(\{\beta_1 \vee \ldots \vee \beta_n\})$, for some $\beta_1, \ldots, \beta_n \in NX$. Thus $\sim \alpha \in dCN(\{\sim \gamma_1 \vee \ldots \vee \sim \gamma_n\})$, where $\gamma_i \in X$. That implies $\alpha Cn(X)$. The proof of (ii) we omits because of its similarity to the previous one.

The notions of system, consistency, completeness, independency and their dual counterparts will be understood as usual.

Remark 1. The authors of [3] pointed out to the possibility of the twofold inequivalent defining the notion of a $Cn^{-1}$ consistent and that of $Cn^{-1}$ complete set. It is easy to show that both ways of defining the respective notions for $dCn$ are equivalent.

Theorem 6. $X$ is a consistent and complete set iff $L - Cn(X) = dCn(NX)$.

In the proof of the Theorem 6 we make use of the following lemmas proved in [3].

Lemma 1. $X$ is a consistent set iff $NCn(X) \subseteq L - Cn(X)$.

Lemma 2. $X$ is a complete set iff $L - Cn(X) \subseteq NCn(X)$.

Theorem 7.

(i) If $Cn(X) = X$, then $dCn(NX) = NX$.

(ii) If $dCn(X) = X$, then $Cn(NX) = NX$.

(iii) If $X$ is a d-consistent set, then $X \cap Cn(\emptyset) = \emptyset$.

(iv) $X$ is consistent set iff $NX$ is d-consistent set.

(v) $X$ is complete set iff $NX$ is d-complete set.

(vi) If $NX$ is a d-independent set, then $X$ is independent set.

(vii) If $NX$ is an independent set, then $X$ is d-independent set.
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References


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